A new construction of vertex algebras and quasi modules for vertex algebras

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Abstract

In this paper, a new construction of vertex algebras from more general vertex operators is given and a notion of quasi module for vertex algebras is introduced and studied. More specifically, a notion of quasi local subset(space) of $\mathrm{Hom}\,(W, W((x)))$ for any vector space W is introduced and studied, generalizing the notion of usual locality in the most possible way, and it is proved that on any maximal quasi local subspace there exists a natural vertex algebra structure and that any quasi local subset of $\mathrm{Hom}\,(W, W((x)))$ generates a vertex algebra. Furthermore, a notion of quasi module for a vertex algebra is introduced and it is proved that W is a quasi module for each of the vertex algebras generated by quasi local subsets of $\mathrm{Hom}\,(W, W((x)))$. A notion of Γ-vertex algebra is also introduced and studied, where Γ is a subgroup of the multiplicative group $\mathbb{C}^{\times}$ of nonzero complex numbers. It is proved that any maximal quasi local subspace of $\mathrm{Hom}\,(W, W((x)))$ is naturally a Γ-vertex algebra and that any quasi local subset of $\mathrm{Hom}\,(W, W((x)))$ generates a Γ-vertex algebra. It is also proved that a Γ-vertex algebra exactly amounts to a vertex algebra equipped with a Γ-module structure which satisfies a certain compatibility condition. Finally, three families of examples are given, involving twisted affine Lie algebras, certain quantum Heisenberg algebras and certain quantum torus Lie algebras.

1 Introduction

Vertex (operator) algebras are often viewed as some kind of generalized algebras, equipped with infinitely many compatible multiplications parameterized by $\mathbb{Z}$. Another viewpoint is that vertex operator algebras are "algebras" of vertex operators, just as classical (associative or Lie) algebras are algebras of linear operators. This particular viewpoint was emphasized in [FLM] and the philosophy was deeply reflected in [Li1] where a theory of representation for a vertex algebra was developed and a general construction of vertex algebras together with modules was given, by using vertex operators.

For a vector space W, a (weak) vertex operator on W ([Li1], [LL]) is an element of $\mathrm{Hom}\,(W, W((x)))$ which is alternatively denoted by $\mathcal{E}(W)$. A subset (subspace) U of

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$\mathcal{E}(W)$ is said to be *local* if for any $a(x), b(x) \in U$, there exists a nonnegative integer k such that

$$(x_1 - x_2)^k a(x_1) b(x_2) = (x_1 - x_2)^k b(x_2) a(x_1). \tag{1.1}$$

It was proved in [Li1] that any maximal local subspace of $\mathcal{E}(W)$ is naturally a vertex algebra with W as a natural module, where the identity operator 1_W is the vacuum vector and for $a(x), b(x) \in \mathcal{E}(W)$, $n \in \mathbb{Z}$,

$$a(x)_n b(x) = \mathrm{Res}_{x_1} \left((x_1 - x)^n a(x_1) b(x) - (-x + x_1)^n b(x) a(x_1) \right). \tag{1.2}$$

Furthermore, it was proved therein that any local subset of $\mathcal{E}(W)$ canonically generates a vertex algebra with W a module. (Notice that Zone Lemma simply implies that any local subset of $\mathcal{E}(W)$ is contained in some maximal local subspace.) If W is taken to be a highest weight module (or more generally a restricted module) for an *untwisted* affine Kac-Moody Lie algebra or the Virasoro algebra, the generating functions form a local subspace of $\mathcal{E}(W)$, so that one has a natural vertex algebra with W as a module. In this way one has vertex (operator) algebras and modules associated with affine Lie algebras (including infinite-dimensional Heisenberg algebras) and the Virasoro algebra. (Historically, vertex operator algebras were constructed differently (see [B1], [DL], [FLM], [FZ]).)

In the theory of Lie algebras, similar to untwisted affine Kac-Moody Lie algebras, there are some other interesting Lie algebras such as *twisted* affine Kac-Moody Lie algebras (see [K1]), Lie algebras of q-pseudodifferential operators on the circle (sin-algebra) (see [G-K-L]) and some quantum torus Lie algebras (see [G-K-L], [BGT]). If W is a "restricted" module for such a Lie algebra, the generating functions, which are elements of $\mathcal{E}(W)$, are not local. Instead, they satisfy the relation

$$p(x_1/x_2) a(x_1) b(x_2) = p(x_1/x_2) b(x_2) a(x_1) \tag{1.3}$$

(cf. (1.1)) for some nonzero polynomial $p(x)$. It has been a fundamental question whether these Lie algebras (through restricted modules) can be associated with some vertex-algebra-like structures in the same (or similar) way as (or to) that untwisted affine Lie algebras are associated with vertex algebras. In [G-K-K], Golenishcheva-Kutuzova and Kac associated these Lie algebras (without central extension) to certain Lie-algebra-like structures called Γ-conformal algebras. (In fact, as it was proved therein, Γ-conformal algebras are Lie algebras acted by Γ by automorphisms.) But the original question still remains open.

In this paper, we give a complete answer to this question by establishing a conceptual result analogous to that of [Li2]. We found that these Lie algebras can also be associated with ordinary vertex algebras in a certain natural way. Namely we proved that the generating functions as elements of $\mathcal{E}(W)$ for a restricted module W still generate an ordinary vertex algebra in a certain *new* and natural way. This is rather unexpected. First, due to the general locality (1.3) of vertex operators we started with, we would expect certain vertex-algebra-like objects which are essentially defined by the general locality. (Motivated by our result we then prove (Proposition 5.1) that vertex-algebra-like

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objects defined by the general locality are ordinary vertex algebras.) Second, the known constructions of vertex algebras from local (twisted) vertex operators (see [Li1], [Li2], cf. [Li3], [GL]) correspond to the notions of (twisted) module or (twisted) representations for vertex algebras, but there is no notion other than that of (twisted) module in the aspect of representation. In reality, those generating functions generate an ordinary vertex algebra, but for this vertex algebra, the space W equipped with the natural action is neither a module nor a twisted module in the usual sense. Then this new construction of vertex algebras from more general vertex operators naturally leads us to a notion of what we call quasi module for a vertex algebra. Quasi modules naturally include modules and certain "deformation" of twisted modules as we shall see through examples. Just as the notions of module and twisted module, this new notion should also be of fundamental importance.

In the following we give an outline of this paper. We first start with what we call quasi local subspaces of $\mathcal{E}(W)$ for an arbitrarily given vector space W. A subset (subspace) U of $\mathcal{E}(W)$ is *quasi local* if for any $a(x), b(x) \in U$, there exists a nonzero polynomial $f(x_1, x_2)$ such that

$$f(x_1, x_2)a(x_1)b(x_2) = f(x_1, x_2)b(x_2)a(x_1). \tag{1.4}$$

(Arguably, this is the most possible generalization of usual locality (1.1).)

We then consider appropriate actions of vertex operators on vertex operators. It is important to point out that in this generality, the actions of vertex operators on vertex operators defined by (1.2) are not appropriate. In fact, as we have seen in [Li1], even in the (fermionic) supercase, (1.2) should be modified; if

$$(x_1 - x_2)^k a(x_1)b(x_2) = -(x_1 - x_2)^k b(x_2)a(x_1) \tag{1.5}$$

for some nonnegative integer k, $a(x)_n b(x)$ should be defined by

$$a(x)_n b(x) = \operatorname{Res}_{x_1} \left((x_1 - x)^n a(x_1)b(x) + (-x + x_1)^n b(x)a(x_1) \right). \tag{1.6}$$

On the other hand, it was proved in [LL] (see also [R]) that for any subspace U of $\mathcal{E}(W)$ which contains the identity operator 1_W and is closed under the action defined by (1.2), U is a vertex algebra if and only if U is local.

In [Li3], motivated by [B2] we have studied suitably defined "compatible" subspaces of $\mathcal{E}(W)$ where compatibility generalizes the usual locality in a certain direction. We introduced a notion of (axiomatic) G_1-vertex algebra and showed that compatible subspaces of $\mathcal{E}(W)$ naturally give rise to G_1-vertex algebras with W as a module. In [Li3], the actions of vertex operators on vertex operations were essentially defined by using the operator product expansion or weak associativity, just as the product for classical associative algebras are defined by the associativity $(ab)w = a(bw)$. Informally, $a(x)_n b(x)$ for $n \in \mathbb{Z}$ were defined in terms of generating function $\mathcal{Y}(a(x), x_0)b(x) = \sum_{n \in \mathbb{Z}} a(x)_n b(x) x_0^{-n-1}$ by

$$\langle w^*, (\mathcal{Y}(a(x), x_0)b(x))w \rangle = \langle w^*, a(x_0 + x)b(x)w \rangle \tag{1.7}$$

for $w^* \in W^*$, $w \in W$. (This was precisely defined in terms of ι-maps.) In this paper, we extend this definition by naturally extending the ι-maps explored in [FLM], [FHL] and [LL].

Note that this definition requires that the product $a(x_1)b(x_2)$ be of a certain form, so that $\mathcal{Y}$ is only a *partial* map on $\mathcal{E}(W) \otimes \mathcal{E}(W)$, unlike the usual case where $\mathcal{Y}$ was defined on the whole space $\mathcal{E}(W) \otimes \mathcal{E}(W)$. Nevertheless, for any quasi local subspace U of $\mathcal{E}(W)$, $\mathcal{Y}$ is a linear map from $U \otimes U$ to $\mathcal{E}(W)((x))$. If U is a maximal quasi local subspace of $\mathcal{E}(W)$, we show that U contains 1_W, $\mathcal{Y}$ maps $U \otimes U$ into $U((x))$ and that $(U, \mathcal{Y}, 1_W)$ carries the structure of a vertex algebra. For any quasi local subset S of $\mathcal{E}(W)$, in view of Zorn lemma, there exists a maximal quasi local subspace U containing S. Consequently, S generates a (canonical) vertex algebra inside $\mathcal{E}(W)$.

There is a new feature of quasi locality. Similar to the map $\mathcal{Y}$ we define a family of (partial) maps $\mathcal{Y}_\alpha$ for $\alpha \in \mathbb{C}^\times$ by

$$\langle w^*, (\mathcal{Y}_\alpha(a(x), x_0)b(x))w \rangle = \langle w^*, a(x_0 + \alpha x)b(x)w \rangle \tag{1.8}$$

for $w^* \in W^*$, $w \in W$. It is proved that if U is a maximal quasi local subspace of $\mathcal{E}(W)$, for any nonzero complex number α, $\mathcal{Y}_\alpha$ maps $U \otimes U$ to $U((x))$, so that U is equipped with a family of linear maps $\mathcal{Y}_\alpha$ for $\alpha \in \mathbb{C}^\times$. (If U is a maximal *local* subspace of $\mathcal{E}(W)$, this is not true in general.) We furthermore obtain a Jacobi-like identity for $(U, \{\mathcal{Y}_\alpha\}, 1_W)$ (Theorem 4.9). Motivated by this and by the notion of Γ-conformal algebra in [G-K-K] we introduce a notion of Γ-vertex algebra, where Γ is any group equipped with a group homomorphism from Γ to $\mathbb{C}^\times$ and we show that a Γ-vertex algebra amounts to an (ordinary) vertex algebra equipped with a Γ-module structure satisfying a certain compatibility condition. In terms of this notion, we have proved (Theorem 4.9) that for any maximal quasi local subspace U of $\mathcal{E}(W)$, $(U, \{\mathcal{Y}_\alpha\}, 1_W)$ carries the structure of a Γ-vertex algebra with Γ any subgroup of $\mathbb{C}^\times$ and that any quasi local subset of $\mathcal{E}(W)$ generates a Γ-vertex algebra.

Another important issue is about the module structure on W for vertex algebras in $\mathcal{E}(W)$, generated by quasi local subsets. Even though we still get ordinary vertex algebras from quasi local vertex operators on W, W equipped with the natural action of those vertex algebras is *not* a module in the usual sense, unlike the usual case [Li1], in which W is naturally a module. Motivated by this, we formulate and study a notion of what we call "quasi module" for a vertex algebra V. A quasi V-module satisfies all the axioms in the definition of the notion of module except that the Jacobi identity axiom is replaced by the axiom that for $u, v \in V$, there exists a nonzero polynomial $f(x_1, x_2)$ (depending on (u, v)) such that the Jacobi identity for (u, v) multiplied by $f(x_1, x_2)$ holds. In terms of this notion, we have that for any vector space W, any maximal quasi local subspace of $\mathcal{E}(W)$ is naturally a vertex algebra with W a quasi module and any quasi local subset of $\mathcal{E}(W)$ generates a vertex algebra with W as a quasi module.

In the last section, as an application of our general results we study three families of examples. In the first family, we consider untwisted affine Lie algebra $\hat{\mathfrak{g}}$ associated to any Lie algebra $\mathfrak{g}$ equipped with a nondegenerate symmetric invariant bilinear form and twisted affine Lie algebra $\hat{\mathfrak{g}}[\sigma]$ associated with an automorphism σ of $\mathfrak{g}$ of (finite) order. We show that on any restricted $\hat{\mathfrak{g}}[\sigma]$-module W of level $\ell \in \mathbb{C}$, there exists a unique quasi module structure for the vertex operator algebra $V_{\hat{\mathfrak{g}}}(\ell, 0)$ (cf. [LL]). It has been known ([Li2], cf. [FLM]) that on any restricted $\hat{\mathfrak{g}}[\sigma]$-module W of level ℓ, there exists a unique σ-twisted module structure for $V_{\hat{\mathfrak{g}}}(\ell, 0)$. This suggests that for a vertex algebra

V, twisted V-modules with respect to finite order automorphisms should be connected with quasi modules. In a sequel [Li5], we shall study the relations between σ-twisted V-modules and quasi V-modules, for a general vertex operator algebra V and for any finite order automorphism σ of V. We conjecture that the category of σ-twisted V-modules is canonically isomorphic to a subcategory of quasi V-modules. In this way, quasi modules unify modules and twisted modules. This connection may shed some light on some difficult problems in the study of orbifold theory (see [DM]).

In the second family, we study a certain "quantum" Heisenberg Lie algebra $\hat{\mathfrak{h}}_q$ associated with a vector space $\mathfrak{h}$ equipped with a nondegenerate symmetric bilinear form and with a nonzero complex number q. We show that any restricted $\hat{\mathfrak{h}}_q$-module is a quasi module for a certain Heisenberg vertex algebra. In the third family, we study certain quantum torus Lie algebras (see [G-K-L], [BGT]) and we show that any restricted module is a quasi module for a certain vertex algebra associated with a certain affine Lie algebra $\hat{\mathfrak{g}}$. It was known (see [BGT]) that some of quantum torus Lie algebras are essentially extended affine Lie algebras (without derivations added). Previously, toroidal extended affine Lie algebras have been related to vertex algebras (see [BBS], [BDT]). It is our belief that all the extended affine Lie algebras (see [AABGP]) can be linked to (general) untwisted affine Lie algebras and then to vertex algebras in a similar way.

While one of our motivations for this paper is to find a solution to the question mentioned earlier, our main motivation is to develop a theory of quantum vertex algebras so that quantum affine algebras in [FJ] will naturally give rise to quantum vertex algebras just as (untwisted) affine Lie algebras naturally give rise to vertex algebras. The quantum vertex operators studied in [FJ] do not satisfy quasi locality, but they are "compatible" in the sense that is defined here, so that the actions of vertex operators on vertex operators have been defined. In several places our setting is more general than we need for this paper. This is exactly for our study on quantum vertex algebras in a sequel [Li4].

The main results of this paper were reported in the workshop "Conformal Field Theory and Vertex Algebras," Osaka, Japan (January 10-12, 2004). We would like to thank Professors Nagatomo and Tsuchiya for invitation and for their hospitality.

This paper is organized as follows: In Section 2, we extend the usual iota maps and we discuss certain associativity and cancelation properties. In Section 3, we introduce certain basic operations on $\mathcal{E}(W)$ and present some basic properties. In Section 4, we study quasi local subspaces of $\mathcal{E}(W)$ and prove the key results. In Section 5, we study vertex algebras and quasi modules. In Section 6, we introduce and study the notions of Γ-vertex algebra and quasi module. In Section 7, we give three families of examples of vertex algebras and quasi modules.

2 Some formal calculus

In this section we define certain iota maps, which generalize the usual iota maps introduced in [FLM], [FHL] and [LL]. We also discuss the subtle issues of associativity and cancelation laws in formal calculus.

First, throughout this paper, $x, y, x_0, x_1, x_2, x_3, \ldots$ are mutually commuting independent formal variables. Vector spaces are considered to be over the field $\mathbb{C}$ of complex numbers, though any algebraically closed field of characteristic zero will work fine.

For a vector space U, $U((x))$ is the space of lower truncated (infinite) integral power series of x with coefficients in U, $U[[x]]$ is the space of (infinite) nonnegative integral power series in x with coefficients in U and $U[[x, x^{-1}]]$ is the space of doubly infinite integral power series in x with coefficients in U. Multi-variable analogues of these spaces are defined in the obvious way. For example, $U[[x_1, x_2, \ldots, x_r]]$ is the space of (infinite) nonnegative integral power series in $x_1, \ldots, x_r$ with coefficients in U and $U((x_1, x_2, \ldots, x_r))$ is the space of lower truncated (infinite) integral power series in $x_1, \ldots, x_r$ with coefficients in U:

$$U((x_1, \ldots, x_r)) = U[[x_1, \ldots, x_r]][x_1^{-1}, \ldots, x_r^{-1}]. \tag{2.1}$$

Remark 2.1. Let F be any field. Then $F((x))$ is a field. In view of this, $\mathbb{C}((x_1))$, $\mathbb{C}((x_1))((x_2)), \ldots, \mathbb{C}((x_1))((x_2)) \cdots ((x_r))$ are fields. Clearly,

$$\mathbb{C}[x_1, \ldots, x_r] \subset \mathbb{C}[[x_1, \ldots, x_r]] \subset \mathbb{C}((x_1, \ldots, x_r)) \subset \mathbb{C}((x_1))((x_2)) \cdots ((x_r)). \tag{2.2}$$

Denote by $\mathbb{C}_*(x_1, \ldots, x_r)$ the localization of $\mathbb{C}[[x_1, \ldots, x_r]]$ at $\mathbb{C}[x_1, \ldots, x_r]^\times$ (the nonzero polynomials):

$$\mathbb{C}_*(x_1, \ldots, x_r) = \mathbb{C}[[x_1, \ldots, x_r]]_{\mathbb{C}[x_1, \ldots, x_r]^\times}. \tag{2.3}$$

The field $\mathbb{C}(x_1, \ldots, x_r)$ of rational functions, i.e., the fraction field of the polynomial ring $\mathbb{C}[x_1, \ldots, x_r]$, is a subring of $\mathbb{C}_*(x_1, \ldots, x_r)$. Since $\mathbb{C}[[x_1, \ldots, x_r]]$ is a subring of $\mathbb{C}((x_1))((x_2)) \cdots ((x_r))$, $\mathbb{C}_*(x_1, \ldots, x_r)$ can be naturally embedded into $\mathbb{C}((x_1))((x_2)) \cdots ((x_r))$. Denote this embedding by $\iota_{x_1, \ldots, x_r}$:

$$\iota_{x_1, \ldots, x_r} : \mathbb{C}_*(x_1, \ldots, x_r) \to \mathbb{C}((x_1))((x_2)) \cdots ((x_r)). \tag{2.4}$$

For any permutation σ on $\{1, \ldots, r\}$, $\iota_{x_{\sigma(1)}, \ldots, x_{\sigma(r)}}$ denotes the embedding

$$\iota_{x_{\sigma(1)}, \ldots, x_{\sigma(r)}} : \mathbb{C}_*(x_1, \ldots, x_r) \to \mathbb{C}((x_{\sigma(1)})) \cdots ((x_{\sigma(r)})). \tag{2.5}$$

In particular, ι_{x_2, x_1} denotes the embedding of $\mathbb{C}_*(x_1, x_2)$ into $\mathbb{C}((x_2))((x_1))$. For example, for any nonzero complex number α and for any integer n,

$$\iota_{x_1, x_2}(x_1 - \alpha x_2)^n = \sum_{i \geq 0} \binom{n}{i} (-\alpha)^i x_1^{n-i} x_2^i = (x_1 - \alpha x_2)^n, \tag{2.6}$$

$$\iota_{x_2, x_1}(x_1 - \alpha x_2)^n = \sum_{i \geq 0} \binom{n}{i} (-\alpha)^{n-i} x_2^{n-i} x_1^i = (-\alpha x_2 + x_1)^n, \tag{2.7}$$

where we are using the usual binomial series expansion conventions. Notice that partial derivative operators $\frac{\partial}{\partial x_i}$ act on both the domains and codomains and that partial derivative operators commute with the iota-maps. Furthermore, all the iota-maps are $\mathbb{C}((x_1, \ldots, x_r))$-homomorphisms and they are identity on $\mathbb{C}((x_1, \ldots, x_r))$.

In formal calculus ([FLM], [FHL], [LL]), associativity and cancelation for products of formal series are subtle issues. Let A be an associative algebra with identity (over $\mathbb{C}$) and let U be an A-module. Let

$$F, G \in A[[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]], \quad H \in U[[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]].$$

The associativity $F(GH) = (FG)H$ does not hold in general and on the other hand, it does hold if all the products GH, FG and FGH exist. As in associative algebra theory, associativity sometimes leads to cancelation. In the following we give a few useful cases.

Lemma 2.2. *Let U be any vector space. (a) Let $f(x_1, x_2) \in U[[x_1^{\pm 1}, x_2^{\pm 1}]]$, $m, n \in \mathbb{Z}$. Then $x_1^m x_2^n f(x_1, x_2) = 0$ if and only if $f(x_1, x_2) = 0$.*
   (b) For $f(x_1, x_2), g(x_1, x_2) \in \mathbb{C}((x_1))((x_2))$ and for $H(x_1, x_2) \in U((x_1))((x_2))$, we have

$$f(x_1, x_2)(g(x_1, x_2)H(x_1, x_2)) = (f(x_1, x_2)g(x_1, x_2))H(x_1, x_2). \tag{2.8}$$

   (c) Let $H(x_1, x_2), K(x_1, x_2) \in U((x_1))((x_2))$. If

$$f(x_1, x_2)H(x_1, x_2) = f(x_1, x_2)K(x_1, x_2) \tag{2.9}$$

for some $0 \neq f(x_1, x_2) \in \mathbb{C}((x_1))((x_2))$, then $H(x_1, x_2) = K(x_1, x_2)$.
   (d) Let $H_1 \in U((x))((x_1))((x_2))$ and $H_2 \in U((x))((x_2))((x_1))$. If there exists a (nonzero) polynomial $f(x_1, x_2)$ with $f(x_1, x_1) \neq 0$ such that

$$f(x_1 + x, x_2 + x)H_1 = f(x_1 + x, x_2 + x)H_2,$$

then $H_1 = H_2$.

Proof. For (a), notice that $U[[x_1^{\pm 1}, x_2^{\pm 1}]]$ is a $\mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}]$-module and $x_1^m x_2^n$ for any $m, n \in \mathbb{Z}$ is invertible in $\mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}]$. Then for $f(x_1, x_2) \in U[[x_1^{\pm 1}, x_2^{\pm 1}]]$, $m, n \in \mathbb{Z}$, $x_1^m x_2^n f(x_1, x_2) = 0$ if and only if $f(x_1, x_2) = 0$.
   Both (b) and (c) follow from the fact that $\mathbb{C}((x_1))((x_2))$ is a field and $U((x_1))((x_2))$ is a vector space over $\mathbb{C}((x_1))((x_2))$.
   For (d), write

$$f(x_1 + x, x_2 + x) = f(x, x) + R(x, x_1, x_2),$$

where $R(x, x_1, x_2) \in \mathbb{C}[x, x_1, x_2]$ with $R(x, 0, 0) = 0$. For each nonnegative integer i, we view $f(x, x)^{-i-1}$ as a formal series (its formal Laurent series expansion at zero) in $\mathbb{C}((x))$. Set

$$g(x, x_1, x_2) = \sum_{i \geq 0} (-1)^i f(x, x)^{-1-i} R(x, x_1, x_2)^i \in \mathbb{C}((x))[[x_1, x_2]]. \tag{2.10}$$

Clearly, $f(x_1 + x, x_2 + x)g(x, x_1, x_2) = 1$. Then it follows from the associativity law. $\square$

Remark 2.3. We recall some fundamental delta-function properties from [FLM], [FHL] and [LL]. For any nonzero complex number α, we have

$$x_0^{-1}\delta\left(\frac{x_1 - \alpha x_2}{x_0}\right) - x_0^{-1}\delta\left(\frac{-\alpha x_2 + x_1}{x_0}\right) = x_1^{-1}\delta\left(\frac{\alpha x_2 + x_0}{x_1}\right). \tag{2.11}$$

For any $g(x_0, x_1, x_2) \in \mathbb{C}((x_0, x_1, x_2))$, we have

$$x_0^{-1}\delta\left(\frac{x_1 - \alpha x_2}{x_0}\right) g(x_0, x_1, x_2) = x_0^{-1}\delta\left(\frac{x_1 - \alpha x_2}{x_0}\right) g(x_0, x_0 + \alpha x_2, x_2), \tag{2.12}$$

$$x_1^{-1}\delta\left(\frac{\alpha x_2 + x_0}{x_1}\right) g(x_0, x_1, x_2) = x_1^{-1}\delta\left(\frac{\alpha x_2 + x_0}{x_1}\right) g(x_0, \alpha x_2 + x_0, x_2). \tag{2.13}$$

Furthermore, if $g(x_0, x_1, x_2)$ involves only nonnegative powers of x_1, using these and (2.11) we get

$$x_0^{-1}\delta\left(\frac{-\alpha x_2 + x_1}{x_0}\right) g(x_0, x_1, x_2) = x_0^{-1}\delta\left(\frac{-\alpha x_2 + x_1}{x_0}\right) g(x_0, \alpha x_2 + x_0, x_2), \tag{2.14}$$

noticing that $g(x_0, \alpha x_2 + x_0, x_2) = g(x_0, x_0 + \alpha x_2, x_2)$.

The following is a reformulation of Proposition 3.4.2 of [LL] with a slightly different proof (cf. [Li1], [Li3]):

Lemma 2.4. *Let U be a vector space, let α be a nonzero complex number and let*

$$\begin{align}
A(x_1, x_2) &\in U((x_1))((x_2)), \tag{2.15}\\
B(x_1, x_2) &\in U((x_2))((x_1)), \tag{2.16}\\
C(x_0, x_2) &\in U((x_2))((x_0)). \tag{2.17}
\end{align}$$

Then

$$x_0^{-1}\delta\left(\frac{x_1 - \alpha x_2}{x_0}\right) A(x_1, x_2) - x_0^{-1}\delta\left(\frac{\alpha x_2 - x_1}{-x_0}\right) B(x_1, x_2)$$

$$= x_1^{-1}\delta\left(\frac{\alpha x_2 + x_0}{x_1}\right) C(x_0, x_2) \tag{2.18}$$

if and only if there exist nonnegative integers k and l such that

$$\begin{align}
(x_1 - \alpha x_2)^k A(x_1, x_2) &= (x_1 - \alpha x_2)^k B(x_1, x_2), \tag{2.19}\\
(x_0 + \alpha x_2)^l A(x_0 + \alpha x_2, x_2) &= (x_0 + \alpha x_2)^l C(x_0, x_2). \tag{2.20}
\end{align}$$

Proof. In the case $\alpha = 1$ it was proved in [Li3]. The general case follows from the special case after the substitution $x_2 \to \alpha^{-1}x_2$. $\qquad\square$

We shall also use the following result (cf. [DLM], [G-K-K], [K2], [Li1], [LL]):

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Lemma 2.5. *Let U be a vector space, let $A(x_1, x_2) \in U[[x_1^{\pm 1}, x_2^{\pm 1}]]$ and let $p(x)$ be a nonzero polynomial. Then*

$$p(x_1/x_2)A(x_1, x_2) = 0 \tag{2.21}$$

if and only if

$$A(x_1, x_2) = \sum_{i=1}^{r} \sum_{j=0}^{k_i - 1} \left(\frac{\alpha_i^{-j}}{j!} \left(\frac{\partial}{\partial x_2} \right)^j x_1^{-1} \delta \left(\frac{\alpha_i x_2}{x_1} \right) \right) B_{i,j}(x_2) \tag{2.22}$$

for some $B_{i,j}(x) \in U[[x, x^{-1}]]$, where $\alpha_1, \ldots, \alpha_r$ are the nonzero (distinct) roots of $p(x)$ with multiplicities $k_1, \ldots, k_r$. Furthermore, these $B_{i,j}(x)$ in (2.22) are uniquely determined by $A(x_1, x_2)$.

Proof. From [Li1] (cf. [LL]) we have

$$(x_1 - x_2)^m \left(\frac{\partial}{\partial x_2} \right)^n x_1^{-1} \delta \left(\frac{x_2}{x_1} \right) = 0 \quad \text{for } m > n \geq 0.$$

For any nonzero complex number β, by changing variable x_2 we have

$$(x_1 - \beta x_2)^m \left(\frac{\partial}{\partial x_2} \right)^n x_1^{-1} \delta \left(\frac{\beta x_2}{x_1} \right) = 0 \quad \text{for } m > n \geq 0. \tag{2.23}$$

Using this we see immediately that (2.22) implies (2.21).

Conversely, assume that (2.21) holds. In view of Lemma 2.2 (d), we may assume that $p(x)$ is monic with $p(0) \neq 0$. Then

$$p(x) = (x - \alpha_1)^{k_1} \cdots (x - \alpha_r)^{k_r}.$$

For $1 \leq i \leq r$, set

$$p_i(x) = p(x)(x - \alpha_i)^{-k_i} \in \mathbb{C}[x]. \tag{2.24}$$

Since $p_1(x), \ldots, p_r(x)$ are relatively prime, there are polynomials $q_1(x), \ldots, q_r(x)$ such that

$$1 = p_1(x)q_1(x) + \cdots + p_r(x)q_r(x). \tag{2.25}$$

We have

$$A(x_1, x_2) = p_1(x_1/x_2)q_1(x_1/x_2)A(x_1, x_2) + \cdots + p_r(x_1/x_2)q_r(x_1/x_2)A(x_1, x_2). \tag{2.26}$$

For $1 \leq i \leq r$, since

$$(x_1/x_2 - \alpha_i)^{k_i} (p_i(x_1/x_2)q_i(x_1/x_2)A(x_1, x_2)) = p(x_1/x_2)q_i(x_1/x_2)A(x_1, x_2) = 0,$$

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from [DLM] (cf. [K2]) we have

$$p_i(x_1/x_2)q_i(x_1/x_2)A(x_1, x_2) = \sum_{j=0}^{k_i-1} \left(\frac{\alpha_i^{-j}}{j!} \left(\frac{\partial}{\partial x_2} \right)^j x_1^{-1} \delta \left(\frac{\alpha_i x_2}{x_1} \right) \right) B_{i,j}(x_2) \qquad (2.27)$$

for some $B_{i,j}(x) \in U[[x, x^{-1}]]$ $(j = 0, \ldots, k_i - 1)$. Now, combining (2.26) with (2.27), we obtain (2.22).

For the uniqueness, let us assume (2.22). Notice that for $s \neq i$, $0 \leq j \leq k_i - 1$, we have

$$p_s(x_1/x_2) \left(\frac{\partial}{\partial x_2} \right)^j x_1^{-1} \delta \left(\frac{\alpha_i x_2}{x_1} \right) = 0,$$

since $p_s(x)$ is a multiple of $(x - \alpha_i)^{k_i}$. Using this (twice) and (2.25), for each i we get

$$
\begin{aligned}
&p_i(x_1/x_2)q_i(x_1/x_2)A(x_1, x_2) \\
=\ &p_i(x_1/x_2)q_i(x_1/x_2) \sum_{j=0}^{k_i-1} \left(\frac{\alpha_i^{-j}}{j!} \left(\frac{\partial}{\partial x_2} \right)^j x_1^{-1} \delta \left(\frac{\alpha_i x_2}{x_1} \right) \right) B_{i,j}(x_2) \\
=\ &\left(1 - \sum_{s \neq i} p_s(x_1/x_2)q_s(x_1/x_2) \right) \sum_{j=0}^{k_i-1} \left(\frac{\alpha_i^{-j}}{j!} \left(\frac{\partial}{\partial x_2} \right)^j x_1^{-1} \delta \left(\frac{\alpha_i x_2}{x_1} \right) \right) B_{i,j}(x_2) \\
=\ &\sum_{j=0}^{k_i-1} \left(\frac{\alpha_i^{-j}}{j!} \left(\frac{\partial}{\partial x_2} \right)^j x_1^{-1} \delta \left(\frac{\alpha_i x_2}{x_1} \right) \right) B_{i,j}(x_2). \qquad (2.28)
\end{aligned}
$$

Then it follows from [Li1] (cf. [LL]) that $B_{i,j}(x)$ are uniquely determined. $\qquad\square$

3 Basic operations on $\mathcal{E}(W)$

In this section, for a vector space W we introduce a natural action of the multiplicative group $\mathbb{C}^\times$ (of nonzero complex numbers) on $\mathcal{E}(W)$, a notion of compatibility for an ordered pair in $\mathcal{E}(W)$ and certain partial operations on $\mathcal{E}(W)$. We then present certain properties analogous to the $\mathcal{D}$-bracket and $\mathcal{D}$-derivative properties for vertex algebras..

Let W be a vector space, fixed throughout this section. Following [LL] we set

$$\mathcal{E}(W) = \mathrm{Hom}\,(W, W((x))). \qquad (3.1)$$

We consider $\mathrm{End}\,W$ as a subspace of $\mathcal{E}(W)$ where each linear operator on W is considered as a constant series in x. The identity operator on W, denoted by 1_W, is a special element of $\mathcal{E}(W)$. Clearly, $\mathcal{E}(W)$ is closed under the formal derivative operator $\frac{d}{dx}$. Set

$$D = \frac{d}{dx}. \qquad (3.2)$$

Just as in calculus, for $a(x) \in \mathcal{E}(W)$, we use $a'(x)$ for the formal derivative of $a(x)$:

$$a'(x) = \frac{d}{dx}a(x) = Da(x). \tag{3.3}$$

For any $\alpha \in \mathbb{C}^\times$ (nonzero complex numbers) and $a(x) \in \mathcal{E}(W)$, it is clear that $a(\alpha x) \in \mathcal{E}(W)$. For $\alpha \in \mathbb{C}^\times$, we define $R_\alpha \in \mathrm{End}\,(\mathcal{E}(W))$ by

$$R_\alpha a(x) = a(\alpha x) \quad \text{for } a(x) \in \mathcal{E}(W) \tag{3.4}$$

(cf. [G-K-K]). Alternatively we have

$$R_\alpha = \alpha^{x\frac{d}{dx}}. \tag{3.5}$$

It is clear that the map $R : \mathbb{C}^\times \to \mathrm{End}\,(\mathcal{E}(W))$ sending α to R_α is a representation of $\mathbb{C}^\times$ on $\mathcal{E}(W)$. We have

$$DR_\alpha = \alpha R_\alpha D \quad \text{for } \alpha \in \mathbb{C}^\times. \tag{3.6}$$

We introduce the following notion of compatibility (cf. [Li3], [B2]):

Definition 3.1. An ordered pair $(a(x), b(x))$ in $\mathcal{E}(W)$ is said to be *compatible* if there exists a nonzero polynomial $f(x_1, x_2)$ such that

$$f(x_1, x_2)a(x_1)b(x_2) \in \mathrm{Hom}\,(W, W((x_1, x_2))). \tag{3.7}$$

A subset(space) U of $\mathcal{E}(W)$ is said to be *pairwise compatible* if every (ordered) pair in U is compatible.

Remark 3.2. Note that this compatibility is more general than the one defined in [Li3] where polynomials $f(x_1, x_2)$ are of the special form $(x_1 - x_2)^k$.

Lemma 3.3. *Let $(a(x), b(x))$ be a compatible (ordered) pair in $\mathcal{E}(W)$ and let $f(x_1, x_2)$ be any nonzero polynomial such that*

$$f(x_1, x_2)a(x_1)b(x_2) \in \mathrm{Hom}\,(W, W((x_1, x_2))). \tag{3.8}$$

For any nonzero complex number α, the expression

$$\iota_{x,x_0}\left(f(x_0 + \alpha x, x)^{-1}\right) x_1^{-1}\delta\left(\frac{\alpha x + x_0}{x_1}\right)(f(x_1, x)a(x_1)b(x)) \tag{3.9}$$

exists in $(\mathrm{End}\,W)[[x^{\pm 1}, x_0^{\pm 1}, x_1^{\pm 1}]]$ and in fact it lies in $(\mathrm{Hom}\,(W, W((x))((x_0)))) [[x_1^{\pm 1}]]$. Furthermore, it is independent of the choice of $f(x_1, x_2)$ in (3.8).

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Proof. Set
$$H(x_1, x_2) = f(x_1, x_2)a(x_1)b(x_2) \in \mathrm{Hom}\,(W, W((x_1, x_2))).$$
From Remark 2.3 we have
$$x_1^{-1}\delta\left(\frac{\alpha x + x_0}{x_1}\right)H(x_1, x) = x_1^{-1}\delta\left(\frac{\alpha x + x_0}{x_1}\right)H(\alpha x + x_0, x),$$
which lies in $(\mathrm{Hom}\,(W, W((x))[[x_0]]))\,[[x_1^{\pm 1}]]$. As $\iota_{x,x_0}\,(f(x_0 + \alpha x, x)^{-1}) \in \mathbb{C}((x))((x_0))$, we have
$$\iota_{x,x_0}\,(f(x_0 + \alpha x, x)^{-1})\,x_1^{-1}\delta\left(\frac{\alpha x + x_0}{x_1}\right)H(\alpha x + x_0, x) \in (\mathrm{Hom}\,(W, W((x))((x_0))))\,[[x_1^{\pm 1}]].$$

Assume that $g(x_1, x_2)$ is another nonzero polynomial such that
$$g(x_1, x_2)a(x_1)b(x_2) \in \mathrm{Hom}\,(W, W((x_1, x_2))).$$

Using Lemma 2.2 we have
$$\iota_{x,x_0}\,\left(f(x_0 + \alpha x, x)^{-1}g(x_0 + \alpha x, x)^{-1}\right)x_1^{-1}\delta\left(\frac{\alpha x + x_0}{x_1}\right)(f(x_1, x)g(x_1, x)a(x_1)b(x))$$
$$= \iota_{x,x_0}\,\left(f(x_0 + \alpha x, x)^{-1}g(x_0 + \alpha x, x)^{-1}\right)x_1^{-1}\delta\left(\frac{\alpha x + x_0}{x_1}\right)f(x_1, x)\,(g(x_1, x)a(x_1)b(x))$$
$$= \iota_{x,x_0}\,\left(f(x_0 + \alpha x, x)^{-1}\right)\iota_{x,x_0}\,\left(g(x_0 + \alpha x, x)^{-1}\right)x_1^{-1}\delta\left(\frac{\alpha x + x_0}{x_1}\right)f(x_0 + \alpha x, x)$$
$$\qquad \cdot (g(x_1, x)a(x_1)b(x))$$
$$= \iota_{x,x_0}\,\left(g(x_0 + \alpha x, x)^{-1}\right)x_1^{-1}\delta\left(\frac{\alpha x + x_0}{x_1}\right)(g(x_1, x)a(x_1)b(x)),$$

noticing that all the assumptions for associativity are satisfied. Symmetrically, we have
$$\iota_{x,x_0}\,\left(f(x_0 + \alpha x, x)^{-1}g(x_0 + \alpha x, x)^{-1}\right)x_1^{-1}\delta\left(\frac{\alpha x + x_0}{x_1}\right)(f(x_1, x)g(x_1, x)a(x_1)b(x))$$
$$= \iota_{x,x_0}\,\left(f(x_0 + \alpha x, x)^{-1}\right)x_1^{-1}\delta\left(\frac{\alpha x + x_0}{x_1}\right)(f(x_1, x)a(x_1)b(x)).$$

It now follows that the expression in (3.9) is independent of the choice of $f(x_1, x_2)$. $\qquad\square$

Now, we introduce the following partial operations on $\mathcal{E}(W)$.

Definition 3.4. Let $(a(x), b(x))$ be a compatible (ordered) pair in $\mathcal{E}(W)$. For $\alpha \in \mathbb{C}^\times$, $n \in \mathbb{Z}$, we define $a(x)_{(\alpha, n)}b(x) \in (\mathrm{End}\,W)[[x, x^{-1}]]$ in terms of generating function
$$\mathcal{Y}_\alpha(a(x), x_0)b(x) = \sum_{n\in\mathbb{Z}}(a(x)_{(\alpha,n)}b(x))x_0^{-n-1} \in (\mathrm{End}\,W)[[x_0^{\pm 1}, x^{\pm 1}]] \qquad (3.10)$$

by

$$\mathcal{Y}_\alpha(a(x), x_0)b(x)$$

$$= \operatorname{Res}_{x_1} \iota_{x,x_0} \left(f(x_0 + \alpha x, x)^{-1} \right) x_1^{-1} \delta \left(\frac{\alpha x + x_0}{x_1} \right) \left(f(x_1, x)a(x_1)b(x) \right) \quad (3.11)$$

$$= \iota_{x,x_0} \left(f(x_0 + \alpha x, x)^{-1} \right) \left(f(x_1, x)a(x_1)b(x) \right)|_{x_1 = \alpha x + x_0}, \quad (3.12)$$

where $f(x_1, x_2)$ is any nonzero polynomial such that

$$f(x_1, x_2)a(x_1)b(x_2) \in \operatorname{Hom}(W, W((x_1, x_2))).$$

We particularly set

$$\mathcal{Y}(a(x), x_0)b(x) = \mathcal{Y}_1(a(x), x_0)b(x), \quad (3.13)$$
$$a(x)_n b(x) = a(x)_{(1,n)} b(x) \quad \text{for } n \in \mathbb{Z}. \quad (3.14)$$

Remark 3.5. In (3.12), we are not allowed to write

$$\left(f(x_1, x)a(x_1)b(x) \right)|_{x_1 = \alpha x + x_0} = f(\alpha x + x_0, x)a(\alpha x + x_0)b(x),$$

since $a(\alpha x + x_0)b(x)$ in general does not exist in $(\operatorname{End} W)[[x^{\pm 1}, x_0^{\pm 1}]]$. On the other hand, $f(\alpha x + x_0, x)a(x_0 + \alpha x)b(x)$ exists. But,

$$\left(f(x_1, x)a(x_1)b(x) \right)|_{x_1 = \alpha x + x_0} \neq \left(f(x_1, x)a(x_1)b(x) \right)|_{x_1 = x_0 + \alpha x}$$
$$\left(= f(x_0 + \alpha x, x)a(x_0 + \alpha x)b(x) \right),$$

unless $f(x_1, x)a(x_1)b(x)$ involves only nonnegative powers of x_1.

First, we have:

Proposition 3.6. *Let $(a(x), b(x))$ be a compatible (ordered) pair in $\mathcal{E}(W)$. Then*

$$a(x)_{(\alpha,n)}b(x) \in \mathcal{E}(W) \ (= \operatorname{Hom}(W, W((x)))) \quad \text{for } \alpha \in \mathbb{C}^\times, \ n \in \mathbb{Z}. \quad (3.15)$$

Let $f(x_1, x_2)$ be any nonzero polynomial such that

$$f(x_1, x_2)a(x_1)b(x_2) \in \operatorname{Hom}(W, W((x_1, x_2))) \quad (3.16)$$

and let k be an integer such that

$$x_0^k \iota_{x_2,x_0} \left(f(x_0 + \alpha x_2, x_2)^{-1} \right) \in \mathbb{C}((x_2))[[x_0]].$$

Then

$$a(x)_{(\alpha,n)}b(x) = 0 \quad \text{for } n \geq k, \quad (3.17)$$

Furthermore, for $w \in W$, let l be a nonnegative integer depending on w such that

$$x_1^l f(x_1, x_2)a(x_1)b(x_2)w \in W[[x_1, x_2]][x_2^{-1}],$$

then

$$(x_0 + \alpha x_2)^l f(x_0 + \alpha x_2, x_2)(\mathcal{Y}_\alpha(a(x_2), x_0)b(x_2))w$$
$$= (x_0 + \alpha x_2)^l f(x_0 + \alpha x_2, x_2)a(x_0 + \alpha x_2)b(x_2)w. \quad (3.18)$$

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Proof. For any nonzero polynomial $f(x_1, x_2)$ satisfying (3.16), by Lemma 3.3 we have

$$(f(x_1, x)a(x_1)b(x))|_{x_1=\alpha x+x_0} \in \mathrm{Hom}\,(W, W((x))[[x_0]]) = (\mathrm{Hom}\,(W, W((x))))[[x_0]].$$

With $\iota_{x,x_0}\,(f(x_0 + \alpha x, x)^{-1}) \in \mathbb{C}((x))((x_0))$, we have

$$\mathcal{Y}_\alpha(a(x), x_0)b(x) \in (\mathrm{Hom}\,(W, W((x))))\,((x_0)),$$

proving (3.15). This also proves (3.17), as

$$
\begin{aligned}
x_0^k \mathcal{Y}_\alpha(a(x), x_0)b(x) &= x_0^k \iota_{x,x_0}\,(f(x_0 + \alpha x, x)^{-1})\,(f(x_1, x)a(x_1)b(x))|_{x_1=\alpha x+x_0}\\
&\in (\mathrm{Hom}\,(W, W((x))))[[x_0]].
\end{aligned}
$$

For $w \in W$, let $\ell \in \mathbb{N}$ (depending on w) such that

$$x_1^l f(x_1, x_2)a(x_1)b(x_2)w \in W[[x_1, x_2]][x_2^{-1}].$$

Then

$$
\begin{aligned}
&\left(x_1^l f(x_1, x_2)a(x_1)b(x_2)w\right)|_{x_1=\alpha x+x_0}\\
={}&\left(x_1^l f(x_1, x_2)a(x_1)b(x_2)w\right)|_{x_1=x_0+\alpha x}\\
={}&(x_0 + \alpha x)^l f(x_0 + \alpha x, x)a(x_0 + \alpha x)b(x)w.
\end{aligned}
$$

Then

$$
\begin{aligned}
&(x_0 + \alpha x)^l f(x_0 + \alpha x, x)\,(\mathcal{Y}_\alpha(a(x), x_0)b(x))\,w\\
={}&\mathrm{Res}\,_{x_1}(x_0 + \alpha x)^l x_1^{-1}\delta\left(\frac{\alpha x + x_0}{x_1}\right)(f(x_1, x)a(x_1)b(x)w)\\
={}&\mathrm{Res}\,_{x_1}x_1^l x_1^{-1}\delta\left(\frac{\alpha x + x_0}{x_1}\right)(f(x_1, x)a(x_1)b(x)w)\\
={}&\left(x_1^l f(x_1, x)a(x_1)b(x)w\right)|_{x_1=\alpha x+x_0}\\
={}&\left(x_1^l f(x_1, x)a(x_1)b(x)w\right)|_{x_1=x_0+\alpha x}\\
={}&(x_0 + \alpha x)^l f(x_0 + \alpha x, x)a(x_0 + \alpha x)b(x)w, \qquad\qquad\qquad (3.19)
\end{aligned}
$$

proving (3.18). Now, the proof is complete. $\qquad\square$

The following result says that we can express $\mathcal{Y}_\alpha(a(x), x_0)b(x)$ in terms of matrix-coefficients and iota-maps:

Lemma 3.7. *Let $(a(x), b(x))$ be a compatible (ordered) pair in $\mathcal{E}(W)$. Then for any nonzero complex number α and for any $w^* \in W^*$, $w \in W$, the formal series*

$$\langle w^*, a(x_0 + \alpha x_2)b(x_2)w\rangle,$$

an element of $\mathbb{C}((x_0))((x_2))$, lies in the image of ι_{x_0,x_2}. Furthermore,

$$\langle w^*, (\mathcal{Y}_\alpha(a(x), x_0)b(x))w\rangle = \iota_{x,x_0}\iota_{x_0,x}^{-1}\langle w^*, a(x_0 + \alpha x)b(x)w\rangle. \qquad\qquad (3.20)$$

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Proof. It follows from (3.18) as in the usual case (cf. [FLM], [FHL], [DL], [LL]). $\square$

The following are analogues of the creation property and the vacuum property in the definition of the notion of vertex algebra:

Proposition 3.8. *For any* $a(x) \in \mathcal{E}(W)$, *the sequences* $(1_W, a(x))$ *and* $(a(x), 1_W)$ *are compatible and for* $\alpha \in \mathbb{C}^\times$ *we have*

$$\mathcal{Y}_\alpha(1_W, x_0)a(x) = a(x), \tag{3.21}$$

$$\mathcal{Y}_\alpha(a(x), x_0)1_W = a(\alpha x + x_0) = e^{\alpha^{-1}x_0\frac{d}{dx}}a(\alpha x) = e^{\alpha^{-1}x_0\frac{d}{dx}}R_\alpha a(x). \tag{3.22}$$

In particular,

$$a(x)_{(\alpha,-1)}1_W = R_\alpha a(x), \tag{3.23}$$

$$a(x)_{-2}1_W \left(= a(x)_{(1,-2)}1_W\right) = \frac{d}{dx}a(x) = Da(x). \tag{3.24}$$

Proof. The compatibility assertion is clear. For $w^* \in W^*$, $w \in W$, using Lemma 3.7 we have

$$\begin{aligned}
\langle w^*, (\mathcal{Y}_\alpha(1_W, x_0)a(x))\, w\rangle &= \iota_{x,x_0}\iota_{x_0,x}^{-1}\langle w^*, 1_W(x_0 + \alpha x)a(x)w\rangle \\
&= \iota_{x,x_0}\iota_{x_0,x}^{-1}\langle w^*, a(x)w\rangle \\
&= \langle w^*, a(x)w\rangle, \tag{3.25}
\end{aligned}$$

proving (3.21). Similarly, for $w^* \in W^*$, $w \in W$, we have

$$\begin{aligned}
\langle w^*, (\mathcal{Y}_\alpha(a(x), x_0)1_W)\, w\rangle &= \iota_{x,x_0}\iota_{x_0,x}^{-1}\langle w^*, a(x_0 + \alpha x)w\rangle \\
&= \langle w^*, a(\alpha x + x_0)w\rangle, \tag{3.26}
\end{aligned}$$

proving (3.22). $\square$

We also have the following properties, analogous to the $\mathcal{D}$-properties for vertex algebras (cf. [LL]):

Proposition 3.9. *Let* $(a(x), b(x))$ *be a compatible (ordered) pair in* $\mathcal{E}(W)$. *Then the ordered pairs* $(a'(x), b(x)), (a(x), b'(x))$ *and* $(a(\alpha x), b(\beta x))$ *for* $\alpha, \beta \in \mathbb{C}^\times$ *are compatible. Furthermore, for* $\alpha, \beta \in \mathbb{C}^\times$, *we have*

$$\mathcal{Y}_\alpha(Da(x), x_0)b(x) = \frac{\partial}{\partial x_0}\mathcal{Y}_\alpha(a(x), x_0)b(x), \tag{3.27}$$

$$D\mathcal{Y}_\alpha(a(x), x_0)b(x) - \mathcal{Y}_\alpha(a(x), x_0)Db(x) = \alpha\frac{\partial}{\partial x_0}\mathcal{Y}_\alpha(a(x), x_0)b(x), \tag{3.28}$$

$$R_\alpha\mathcal{Y}_\beta(a(x), x_0)b(x) = \mathcal{Y}_{\alpha\beta}(a(x), x_0)R_\alpha b(x), \tag{3.29}$$

$$\mathcal{Y}_\alpha(R_\beta a(x), \beta^{-1}x_0)b(x) = \mathcal{Y}_{\alpha\beta}(a(x), x_0)b(x), \tag{3.30}$$

In particular,

$$\mathcal{Y}_\alpha(a(x), x_0)b(x) = R_\alpha\mathcal{Y}_1(a(x), x_0)R_{\alpha^{-1}}b(x) = \mathcal{Y}_1(R_\alpha a(x), \alpha^{-1}x_0). \tag{3.31}$$

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Proof. Let $0 \neq f(x_1, x_2) \in \mathbb{C}[x_1, x_2]$ such that

$$f(x_1, x_2)a(x_1)b(x_2) \in \text{Hom}\,(W, W((x_1, x_2))).$$

We immediately have

$$f(\alpha x_1, \beta x_2)a(\alpha x_1)b(\beta x_2) \in \text{Hom}\,(W, W((x_1, x_2))).$$

We also have

$$f(x_1, x_2)^2 a(x_1)b(x_2), \quad f(x_1, x_2)f_{x_1}(x_1, x_2)a(x_1)b(x_2) \in \text{Hom}\,(W, W((x_1, x_2))). \quad (3.32)$$

Then

$$f(x_1, x_2)^2 a'(x_1)b(x_2) = \frac{\partial}{\partial x_2}\left(f(x_1, x_2)^2 a'(x_1)b(x_2)\right) - 2f(x_1, x_2)f_{x_1}(x_1, x_2)a(x_1)b(x_2)$$
$$\in \quad \text{Hom}\,(W, W((x_1, x_2))). \quad (3.33)$$

This proves the compatibility assertions. For simplicity, we locally use ∂_x for $\frac{\partial}{\partial x}$ in this proof. Let $w^* \in W^*$, $w \in W$. Using Lemma 3.7 we have

$$
\begin{aligned}
\langle w^*, (\mathcal{Y}_\alpha(a'(x), x_0)b(x))w \rangle &= \iota_{x,x_0}\iota_{x_0,x}^{-1}\langle w^*, a'(x_0 + \alpha x)b(x)w \rangle \\
&= \iota_{x,x_0}\iota_{x_0,x}^{-1}\partial_{x_0}\langle w^*, a(x_0 + \alpha x)b(x)w \rangle \\
&= \partial_{x_0}\iota_{x,x_0}\iota_{x_0,x}^{-1}\langle w^*, a(x_0 + \alpha x)b(x)w \rangle \\
&= \partial_{x_0}\langle w^*, (\mathcal{Y}_\alpha(a(x), x_0)b(x))w \rangle, \quad (3.34)
\end{aligned}
$$

proving (3.27). Similarly, we have

$$
\begin{aligned}
&\langle w^*, (D\mathcal{Y}_\alpha(a(x), x_0)b(x))w \rangle \\
&= \partial_x\langle w^*, (\mathcal{Y}_\alpha(a(x), x_0)b(x))w \rangle \\
&= \partial_x\iota_{x,x_0}\iota_{x_0,x}^{-1}\langle w^*, a(x_0 + \alpha x)b(x)w \rangle \\
&= \iota_{x,x_0}\iota_{x_0,x}^{-1}\langle w^*, \alpha a'(x_0 + \alpha x)b(x) + a(x_0 + \alpha x)b'(x)w \rangle \\
&= \iota_{x,x_0}\iota_{x_0,x}^{-1}\alpha\partial_{x_0}\langle w^*, a(x_0 + \alpha x)b(x)w \rangle + \iota_{x,x_0}\iota_{x_0,x}^{-1}\langle w^*, a(x_0 + \alpha x)b'(x)w \rangle \\
&= \alpha\partial_{x_0}\langle w^*, (\mathcal{Y}_\alpha(a(x), x_0)b(x))w \rangle + \langle w^*, (\mathcal{Y}_\alpha(a(x), x_0)b'(x))w \rangle, \quad (3.35)
\end{aligned}
$$

proving (3.28). For $\beta \in \mathbb{C}^\times$, let R_β naturally act on $\mathcal{E}(W)[[x_0^{\pm 1}]]$. We have

$$
\begin{aligned}
&\langle w^*, (R_\alpha\mathcal{Y}_\beta(a(x), x_0)b(x))w \rangle \\
&= \alpha^{x\frac{d}{dx}}\langle w^*, (\mathcal{Y}_\beta(a(x), x_0)b(x))w \rangle \\
&= \alpha^{x\frac{d}{dx}}\iota_{x,x_0}\iota_{x_0,x}^{-1}\langle w^*, a(x_0 + \beta x)b(x)w \rangle \\
&= \iota_{x,x_0}\iota_{x_0,x}^{-1}\langle w^*, a(x_0 + \alpha\beta x)b(\alpha x)w \rangle \\
&= \iota_{x,x_0}\iota_{x_0,x}^{-1}\langle w^*, a(x_0 + \alpha\beta x)(R_\alpha b(x))w \rangle \\
&= \langle w^*, (\mathcal{Y}_{\alpha\beta}(a(x), x_0)R_\alpha b(x))w \rangle. \quad (3.36)
\end{aligned}
$$

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This proves (3.29). The identity (3.30) also holds, since

$$
\begin{aligned}
& \langle w^*, (\mathcal{Y}_\alpha(R_\beta(a(x), \beta^{-1}x_0)b(x))w \rangle \\
= \ & \iota_{x,x_0}\iota_{x_0,x}^{-1}\langle w^*, (R_\beta a)(\beta^{-1}x_0 + \alpha x)b(x)w \rangle \\
= \ & \iota_{x,x_0}\iota_{x_0,x}^{-1}\langle w^*, a(x_0 + \alpha\beta x)b(x))w \rangle \\
= \ & \langle w^*, (\mathcal{Y}_{\alpha\beta}(a(x), x_0)b(x))w \rangle.
\end{aligned}
\tag{3.37}
$$

Specializing α, β appropriately we obtain the particular case; specialize $\beta = 1$ in (3.29) to get the first equality and substitute (α, β) with $(1, \alpha)$ in (3.30) to get the outside equality. $\qquad\square$

The following result gives Jacobi-like identities:

Proposition 3.10. *Let* $a(x), b(x) \in \mathcal{E}(W)$. *Suppose that there exist*

$$
0 \neq f(x_1, x_2) \in \mathbb{C}[x_1, x_2], \ K(x_2, x_1) \in \mathrm{Hom}\,(W, W((x_2))((x_1)))
$$

such that

$$
f(x_1, x_2)a(x_1)b(x_2) = f(x_1, x_2)K(x_2, x_1).
\tag{3.38}
$$

Then $(a(x), b(x))$ *is compatible and for any* $\alpha \in \mathbb{C}^\times$,

$$
\begin{aligned}
& x_1^{-1}\delta\left(\frac{\alpha x + x_0}{x_1}\right) f(x_1, x)\mathcal{Y}_\alpha(a(x), x_0)b(x) \\
= \ & x_0^{-1}\delta\left(\frac{x_1 - \alpha x}{x_0}\right) f(x_1, x)a(x_1)b(x) - x_0^{-1}\delta\left(\frac{\alpha x - x_1}{-x_0}\right) f(x_1, x)K(x, x_1). \tag{3.39}
\end{aligned}
$$

Proof. Noticing that the expression on the left hand side of (3.38) lies in $\mathrm{Hom}\,(W, W((x_1))((x_2)))$ and the expression on the right hand side lies in $\mathrm{Hom}\,(W, W((x_2))((x_1)))$, we have

$$
f(x_1, x_2)a(x_1)b(x_2) \in \mathrm{Hom}\,(W, W((x_1, x_2))),
$$

proving that $(a(x), b(x))$ is compatible. Let $w \in W$ be arbitrarily fixed. There exists $l \in \mathbb{N}$ such that

$$
x_1^l f(x_1, x_2)a(x_1)b(x_2)w \in W[[x_1, x_2]][x_2^{-1}].
$$

By Proposition 3.6 we have

$$
\begin{aligned}
& (x_0 + \alpha x)^l f(x_0 + \alpha x, x) \left(\mathcal{Y}_\alpha(a(x), x_0)b(x)\right) w \\
= \ & (x_0 + \alpha x)^l f(x_0 + \alpha x, x)a(x_0 + \alpha x)b(x)w.
\end{aligned}
\tag{3.40}
$$

By Lemma 2.4 with

$$
A = f(x_1, x_2)a(x_1)b(x_2)w, \ \ B = f(x_1, x_2)K(x_2, x_1)w
$$

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and
$$C = f(x_0 + \alpha x, x)a(x_0 + \alpha x)b(x)w$$

we have

$$x_1^{-1}\delta\left(\frac{\alpha x + x_0}{x_1}\right) f(x_0 + \alpha x, x)\left(\mathcal{Y}_\alpha(a(x), x_0)b(x)\right)w$$
$$= x_0^{-1}\delta\left(\frac{x_1 - \alpha x}{x_0}\right) f(x_1, x)a(x_1)b(x)w - x_0^{-1}\delta\left(\frac{-\alpha x + x_1}{x_0}\right) f(x_1, x)K(x, x_1)w. \quad (3.41)$$

Since (by (2.14))

$$x_1^{-1}\delta\left(\frac{\alpha x + x_0}{x_1}\right) f(x_0 + \alpha x, x) = x_1^{-1}\delta\left(\frac{\alpha x + x_0}{x_1}\right) f(x_1, x),$$

we immediately have (3.39). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We also have the following result:

Proposition 3.11. *Let* $a(x), b(x) \in \mathcal{E}(W)$ *be such that*

$$p(x_1/x_2)a(x_1)b(x_2) = p(x_1/x_2)K(x_2, x_1) \quad (3.42)$$

for some nonzero polynomial $p(x)$ *and for some* $K(x_2, x_1) \in \mathrm{Hom}\,(W, W((x_2))((x_1)))$. *Then* $(a(x), b(x))$ *is compatible and*

$$a(x_1)b(x_2) - K(x_2, x_1) = \sum_{\alpha \in \mathbb{C}^\times} \mathrm{Res}\,_{x_0}x_1^{-1}\delta\left(\frac{\alpha x_2 + x_0}{x_1}\right) \mathcal{Y}_\alpha(a(x_2), x_0)b(x_2), \quad (3.43)$$

which is a finite sum over only nonzero roots α *of* $p(x)$.

Proof. The ordered pair $(a(x), b(x))$ is compatible because

$$x_2^{\deg p(x)}p(x_1/x_2)a(x_1)b(x_2) = x_2^{\deg p(x)}p(x_1/x_2)K(x_2, x_1)$$

with $0 \neq x_2^{\deg p(x)}p(x_1/x_2) \in \mathbb{C}[x_1, x_2]$. If $p(x) = x^k q(x)$ where $k \in \mathbb{N}$, $q(x) \in \mathbb{C}[x]$, then $q(x_1/x_2)a(x_1)b(x_2) = q(x_1/x_2)K(x_2, x_1)$. In view of this, we may assume that $p(0) \neq 0$. Let $\alpha_1, \ldots, \alpha_r$ be the distinct (nonzero) roots of $p(x)$ of multiplicities $k_1, \ldots, k_r$. For $1 \leq i \leq r$, let $p_i(x)$ be the (unique) polynomial such that

$$p(x) = p_i(x)(x - \alpha_i)^{k_i} \quad (\text{with}\ \ p_i(\alpha_i) \neq 0). \quad (3.44)$$

Notice that (3.39) with $f(x_1, x_2)$ being replaced by $p(x_1/x_2)$ holds. For $1 \leq i \leq r$, multiplying by $(x_2/x_0)^{k_i}$ and then taking $\mathrm{Res}\,_{x_0}$ (of (3.39)) we get

$$p_i(x_1/x_2)(a(x_1)b(x_2) - K(x_2, x_1))$$
$$= \mathrm{Res}\,_{x_0}p_i(x_1/x_2)x_1^{-1}\delta\left(\frac{\alpha_i x_2 + x_0}{x_1}\right) \mathcal{Y}_{\alpha_i}(a(x_2), x_0)b(x_2). \quad (3.45)$$

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Since $p_1(x), \ldots, p_r(x)$ are relatively prime, there are polynomials $q_1(x), \ldots, q_r(x)$ such that

$$1 = q_1(x)p_1(x) + \cdots + q_r(x)p_r(x). \tag{3.46}$$

Using this and (3.45) we get

$$
\begin{aligned}
& a(x_1)b(x_2) - K(x_2, x_1) \\
= {} & \sum_{i=1}^{r} q_i(x_1/x_2)p_i(x_1/x_2)(a(x_1)b(x_2) - K(x_2, x_1)) \\
= {} & \sum_{i=1}^{r} \operatorname{Res}_{x_0} q_i(x_1/x_2)p_i(x_1/x_2)x_1^{-1}\delta\left(\frac{\alpha_i x_2 + x_0}{x_1}\right) \mathcal{Y}_{\alpha_i}(a(x_2), x_0)b(x_2). \tag{3.47}
\end{aligned}
$$

In view of Proposition 3.6 we have

$$x_0^{k_i}\mathcal{Y}_{\alpha_i}(a(x), x_0)b(x) \in \mathcal{E}(W)[[x_0]] \quad \text{for } i = 1, \ldots, r. \tag{3.48}$$

For $i \neq j$, since $p_i(\alpha_j + x_0/x) \in x_0^{k_j}\mathbb{C}[x, x^{-1}, x_0]$, using (2.13) and (3.48) we get

$$
\begin{aligned}
& \operatorname{Res}_{x_0} q_i(x_1/x_2)p_i(x_1/x_2)x_1^{-1}\delta\left(\frac{\alpha_j x_2 + x_0}{x_1}\right) \mathcal{Y}_{\alpha_j}(a(x_2), x_0)b(x_2) \\
= {} & \operatorname{Res}_{x_0} q_i(\alpha_j + x_0/x_2)p_i(\alpha_j + x_0/x_2)x_1^{-1}\delta\left(\frac{\alpha_j x_2 + x_0}{x_1}\right) \mathcal{Y}_{\alpha_j}(a(x_2), x_0)b(x_2) \\
= {} & 0. \tag{3.49}
\end{aligned}
$$

Using (3.46) and (3.49) we get

$$
\begin{aligned}
& \operatorname{Res}_{x_0} q_j(x_1/x_2)p_j(x_1/x_2)x_1^{-1}\delta\left(\frac{\alpha_j x_2 + x_0}{x_1}\right) \mathcal{Y}_{\alpha_j}(a(x_2), x_0)b(x_2) \\
= {} & \sum_{i=1}^{r} \operatorname{Res}_{x_0} q_i(x_1/x_2)p_i(x_1/x_2)x_1^{-1}\delta\left(\frac{\alpha_j x_2 + x_0}{x_1}\right) \mathcal{Y}_{\alpha_j}(a(x_2), x_0)b(x_2) \\
= {} & \operatorname{Res}_{x_0} x_1^{-1}\delta\left(\frac{\alpha_j x_2 + x_0}{x_1}\right) \mathcal{Y}_{\alpha_j}(a(x_2), x_0)b(x_2), \tag{3.50}
\end{aligned}
$$

so that

$$a(x_1)b(x_2) - K(x_2, x_1) = \sum_{i=1}^{r} \operatorname{Res}_{x_0} x_1^{-1}\delta\left(\frac{\alpha_i x_2 + x_0}{x_1}\right) \mathcal{Y}_{\alpha_i}(a(x_2), x_0)b(x_2). \tag{3.51}$$

Since $\mathcal{Y}_\alpha(a(x), x_0)b(x) \in \mathcal{E}(W)[[x_0]]$ for $\alpha \neq \alpha_1, \ldots, \alpha_r$, we immediately have (3.43). $\qquad\square$

The following is an important consequence of Proposition 3.11:

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Corollary 3.12. *Let* $a(x), b(x) \in \mathcal{E}(W)$. *Suppose that there are* $0 \neq p(x) \in \mathbb{C}[x]$, $f_i(x) \in \mathbb{C}((x))$ *and* $c_i(x), d_i(x) \in \mathcal{E}(W)$ *for* $i = 1, \ldots, r$ *such that*

$$p(x_1/x_2)a(x_1)b(x_2) = p(x_1/x_2) \sum_{i=1}^{r} f_i(x_1/x_2)c_i(x_2)d_i(x_1). \tag{3.52}$$

Then $(a(x), b(x))$ *is compatible and*

$$a(x_1)b(x_2) - \sum_{i=1}^{r} f_i(x_1/x_2)c_i(x_2)d_i(x_1)$$

$$= \sum_{\alpha \in \mathbb{C}^\times} \text{Res}_{x_0} x_1^{-1} \delta\left(\frac{\alpha x_2 + x_0}{x_1}\right) \mathcal{Y}_\alpha(a(x_2), x_0)b(x_2), \tag{3.53}$$

which is a finite sum over only nonzero roots of $p(x)$. *In particular, if*

$$p(x_1/x_2)a(x_1)b(x_2) = p(x_1/x_2)b(x_2)a(x_1) \tag{3.54}$$

for some nonzero polynomial $p(x)$, *then* $(a(x), b(x))$ *is compatible and*

$$[a(x_1), b(x_2)] = \sum_{\alpha \in \mathbb{C}^\times} \text{Res}_{x_0} x_1^{-1} \delta\left(\frac{\alpha x_2 + x_0}{x_1}\right) \mathcal{Y}_\alpha(a(x_2), x_0)b(x_2), \tag{3.55}$$

which is a finite sum over only nonzero roots of $p(x)$.

In the same spirit we have the following result which will be very useful in determining the structure of certain vertex algebras later:

Proposition 3.13. *Let* W *be a vector space as before and let* $a(x), b(x) \in \mathcal{E}(W)$. *Assume that there exists* $K(x_2, x_1) \in \text{Hom}\,(W, W((x_2))((x_1)))$ *such that*

$$a(x_1)b(x_2) - K(x_2, x_1) = \sum_{i=1}^{r} \sum_{j=0}^{k_i} c_{i,j}(x_2) \frac{1}{j!}\left(\alpha_i^{-1} \frac{\partial}{\partial x_2}\right)^j x_1^{-1} \delta\left(\frac{\alpha_i x_2}{x_1}\right), \tag{3.56}$$

where $\alpha_1, \ldots, \alpha_r$ *are finitely many distinct nonzero complex numbers,* k_i *are nonnegative integers and* $c_{i,j}(x) \in \mathcal{E}(W)$. *Then* $(a(x), b(x))$ *is compatible and*

$$a(x)_{(\alpha,n)}b(x) = 0 \qquad \text{for } n \geq 0, \ \alpha \neq \alpha_1, \ldots, \alpha_r, \tag{3.57}$$
$$a(x)_{(\alpha_i,n)}b(x) = c_{i,n}(x) \quad \text{for } 0 \leq n \leq k_i, \tag{3.58}$$
$$a(x)_{(\alpha_i,n)}b(x) = 0 \qquad \text{for } n > k_i. \tag{3.59}$$

Proof. From (3.56) we have

$$(x_1 - \alpha_1 x_2)^m \cdots (x_1 - \alpha_r x_2)^m (a(x_1)b(x_2) - K(x_2, x_1)) = 0$$

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for any nonnegative integer $m \geq \max\{k_1+1, \ldots, k_r+1\}$. By Proposition 3.11, $(a(x), b(x))$ is compatible and

$$
\begin{aligned}
& a(x_1)b(x_2) - K(x_2, x_1) \\
= {} & \sum_{i=1}^{r} \operatorname{Res}_{x_0} x_1^{-1} \delta\left(\frac{\alpha_i x_2 + x_0}{x_1}\right) \mathcal{Y}_{\alpha_i}(a(x_2), x_0) b(x_2) \\
= {} & \sum_{i=1}^{r} \sum_{j=0}^{m} \operatorname{Res}_{x_0} a(x_2)_{(\alpha_i, j)} b(x_2) \frac{1}{j!} \left(\alpha_i^{-1} \frac{\partial}{\partial x_2}\right)^j x_1^{-1} \delta\left(\frac{\alpha_i x_2}{x_1}\right). \qquad (3.60)
\end{aligned}
$$

Combining (3.56) with (3.60), by Lemma 2.5 we have all the rest assertions. $\qquad\square$

Remark 3.14. Notice that the defined maps $(a(x), b(x)) \mapsto a(x)_{(\alpha, n)} b(x)$ are just partial operations on $\mathcal{E}(W)$. If U is a pairwise compatible subspace of $\mathcal{E}(W)$ (in the sense that every (ordered) pair in U is compatible), then these maps are bilinear maps from $U \otimes U$ to $\mathcal{E}(W)$. Furthermore, if U contains 1_W and if U is closed under all these bilinear operations $(a(x), b(x)) \mapsto a(x)_{(\alpha, n)} b(x)$ for $\alpha \in \mathbb{C}^\times$, $n \in \mathbb{Z}$, we would like to know the algebraic structure that $(U, \{\mathcal{Y}_\alpha\}, 1_W)$ carries.

4 Algebraic structures on quasi-local subspaces of $\mathcal{E}(W)$

In this section, we investigate the algebraic structures on "quasi local" subspaces of $\mathcal{E}(W)$ for any given vector space W. We show that for any quasi local subset S of $\mathcal{E}(W)$, there exists a "closed" quasi local subspace U which contains S and 1_W and we furthermore show that $(U, \mathcal{Y}_1, 1_W)$ carries the structure of a vertex algebra. For a general "closed" quasi local subspace U we establish a Jacobi-like identity for the adjoint vertex operators $\mathcal{Y}_\alpha(a(x), x_0)$ for $\alpha \in \mathbb{C}^\times$. The key and most difficult results are Propositions 4.3 and 4.8, whose proofs involve rather complicated formal calculus. The main results are summarized in Theorem 4.9.

As in Section 3, let W be a vector space fixed throughout this section. The following notion generalizes the notion of mutual locality ([DL], [Li2], cf. [G-K-K]):

Definition 4.1. Formal series $a(x), b(x) \in \mathcal{E}(W)$ are said to be *mutually quasi local* if there exists a nonzero polynomial $f(x_1, x_2)$ such that

$$
f(x_1, x_2)a(x_1)b(x_2) = f(x_1, x_2)b(x_2)a(x_1). \qquad (4.1)
$$

A subset (subspace) U of $\mathcal{E}(W)$ is said to be *quasi-local* if any $a(x), b(x) \in U$ (where $a(x)$ and $b(x)$ may be the same) are mutually quasi local. Clearly, any quasi-local subset of $\mathcal{E}(W)$ (linearly) spans a quasi local subspace and any quasi-local subset of $\mathcal{E}(W)$ is contained in some maximal quasi local subspace of $\mathcal{E}(W)$.

We also have the following notion (cf. [G-K-K]):

Definition 4.2. Let Γ be a subgroup of $\mathbb{C}^\times$. Formal series $a(x), b(x) \in \mathcal{E}(W)$ are said to be *mutually Γ-local* if there exists a (nonzero) polynomial

$$f(x_1, x_2) \in \langle (x_1 - \alpha x_2) \mid \alpha \in \Gamma \rangle \subset \mathbb{C}[x_1, x_2] \tag{4.2}$$

such that (4.1) holds. The notion of Γ-local subset(space) of $\mathcal{E}(W)$ is defined in the obvious way.

The following is the first key result:

Proposition 4.3. *Let $a(x), b(x), c(x) \in \mathcal{E}(W)$. Assume that*

$$f(x_1, x_2)a(x_1)b(x_2) = f(x_1, x_2)b(x_2)a(x_1), \tag{4.3}$$
$$g(x_1, x_2)a(x_1)c(x_2) = \tilde{g}(x_1, x_2)c(x_2)a(x_1), \tag{4.4}$$
$$h(x_1, x_2)b(x_1)c(x_2) = \tilde{h}(x_1, x_2)c(x_2)b(x_1), \tag{4.5}$$

where $f(x, y), g(x, y), \tilde{g}(x, y), h(x, y), \tilde{h}(x, y)$ are nonzero polynomials. Then for any $\alpha \in \mathbb{C}^\times$ and for any $n \in \mathbb{Z}$, there exists $k \in \mathbb{N}$ (depending on n) such that

$$f(x_3, \alpha x)^k g(x_3, x)a(x_3)(b(x)_{(\alpha, n)}c(x)) = f(x_3, \alpha x)^k \tilde{g}(x_3, x)(b(x)_{(\alpha, n)}c(x))a(x_3). \tag{4.6}$$

Proof. Let $n \in \mathbb{Z}$ be arbitrarily fixed. Let k be a nonnegative integer depending on n such that

$$x_0^{k+n} \iota_{x,x_0}(h(\alpha x + x_0, x)^{-1}) \in \mathbb{C}((x))[[x_0]]. \tag{4.7}$$

(Recall that $\iota_{x,x_0}(h(\alpha x + x_0, x)^{-1}) \in \mathbb{C}((x))((x_0))$.) Noticing that $\left(\frac{\partial}{\partial x_1}\right)^i f(x_3, x_1)^k$ is a multiple of $f(x_3, x_1)$ for $0 \le i < k$, from (4.3) we have

$$\left(\left(\frac{\partial}{\partial x_1}\right)^i f(x_3, x_1)^k\right) a(x_3)b(x_1) = \left(\left(\frac{\partial}{\partial x_1}\right)^i f(x_3, x_1)^k\right) b(x_1)a(x_3) \quad \text{for } 0 \le i < k. \tag{4.8}$$

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Using (4.7), (4.8) and (4.4) we obtain

$$f(x_3, \alpha x)^k g(x_3, x)a(x_3)\left(b(x)_{(\alpha,n)}c(x)\right)$$

$$= \operatorname{Res}_{x_0} x_0^n f(x_3, \alpha x)^k g(x_3, x)a(x_3)\left(\mathcal{Y}_\alpha(b(x), x_0)c(x)\right)$$

$$= \operatorname{Res}_{x_0}\operatorname{Res}_{x_1} x_0^n f(x_3, \alpha x)^k g(x_3, x)a(x_3)\iota_{x,x_0}(h(\alpha x + x_0, x)^{-1})$$

$$\cdot x_1^{-1}\delta\left(\frac{\alpha x + x_0}{x_1}\right)(h(x_1, x)b(x_1)c(x))$$

$$= \operatorname{Res}_{x_0}\operatorname{Res}_{x_1} x_0^n f(x_3, x_1 - x_0)^k g(x_3, x)a(x_3)\iota_{x,x_0}(h(\alpha x + x_0, x)^{-1})$$

$$\cdot x_1^{-1}\delta\left(\frac{\alpha x + x_0}{x_1}\right)(h(x_1, x)b(x_1)c(x))$$

$$= \operatorname{Res}_{x_0}\operatorname{Res}_{x_1} x_0^n \left(e^{-x_0\frac{\partial}{\partial x_1}}f(x_3, x_1)^k\right)g(x_3, x)a(x_3)\iota_{x,x_0}(h(\alpha x + x_0, x)^{-1})$$

$$\cdot x_1^{-1}\delta\left(\frac{\alpha x + x_0}{x_1}\right)(h(x_1, x)b(x_1)c(x))$$

$$= \operatorname{Res}_{x_0}\operatorname{Res}_{x_1}\sum_{i=0}^{k-1}\frac{(-1)^i}{i!}x_0^{n+i}\left(\left(\frac{\partial}{\partial x_1}\right)^i f(x_3, x_1)^k\right)g(x_3, x)a(x_3)\iota_{x,x_0}(h(\alpha x + x_0, x)^{-1})$$

$$\cdot x_1^{-1}\delta\left(\frac{\alpha x + x_0}{x_1}\right)(h(x_1, x)b(x_1)c(x))$$

$$= \operatorname{Res}_{x_0}\operatorname{Res}_{x_1}\sum_{i=0}^{k-1}\frac{(-1)^i}{i!}x_0^{n+i}\left(\left(\frac{\partial}{\partial x_1}\right)^i f(x_3, x_1)^k\right)\tilde{g}(x_3, x)\iota_{x,x_0}(h(\alpha x + x_0, x)^{-1})$$

$$\cdot x_1^{-1}\delta\left(\frac{\alpha x + x_0}{x_1}\right)(h(x_1, x)b(x_1)c(x))a(x_3)$$

$$= \operatorname{Res}_{x_0}\operatorname{Res}_{x_1} x_0^n \left(e^{-x_0\frac{\partial}{\partial x_1}}f(x_3, x_1)^k\right)\tilde{g}(x_3, x)\iota_{x,x_0}(h(\alpha x + x_0, x)^{-1})$$

$$\cdot x_1^{-1}\delta\left(\frac{\alpha x + x_0}{x_1}\right)(h(x_1, x)b(x_1)c(x))a(x_3)$$

$$= \operatorname{Res}_{x_0}\operatorname{Res}_{x_1} x_0^n f(x_3, x_1 - x_0)^k \tilde{g}(x_3, x)\iota_{x,x_0}(h(\alpha x + x_0, x)^{-1})$$

$$\cdot x_1^{-1}\delta\left(\frac{\alpha x + x_0}{x_1}\right)(h(x_1, x)b(x_1)c(x))a(x_3)$$

$$= \operatorname{Res}_{x_0} x_0^n f(x_3, \alpha x)^k \tilde{g}(x_3, x)\left(\mathcal{Y}_\alpha(b(x), x_0)c(x)\right)a(x_3)$$

$$= f(x_3, \alpha x)^k \tilde{g}(x_3, x)\left(b(x)_{(\alpha,n)}c(x)\right)a(x_3). \tag{4.9}$$

This completes the proof. $\qquad\square$

Recall Remark 3.14 that if U is a pairwise compatible subspace of $\mathcal{E}(W)$, we have linear maps $\mathcal{Y}_\alpha$ for $\alpha \in \mathbb{C}^\times$ from U to $\operatorname{Hom}(U, \mathcal{E}(W)((x)))$. In view of Propositions 3.6 and 3.10, any quasi local subspace U of $\mathcal{E}(W)$ is pairwise compatible so that we have these linear maps.

Definition 4.4. Let Γ be a subgroup of $\mathbb{C}^{\times}$. A pairwise compatible subspace U of $\mathcal{E}(W)$ is said to be $\mathcal{Y}_{\Gamma}$-*closed* if

$$a(x)_{(\alpha,n)}b(x) \in U \quad \text{for } a(x), b(x) \in U, \ \alpha \in \Gamma, \ , n \in \mathbb{Z}. \tag{4.10}$$

In the case $\Gamma = \{1\}$, we say U is $\mathcal{Y}_1$-closed.

Using Proposition 4.3 we have (cf. [LL]):

Corollary 4.5. *Let W be any vector space. Every maximal quasi local subspace U of $\mathcal{E}(W)$ contains 1_W and it is $\mathcal{Y}_{\Gamma}$-closed for any subgroup Γ of $\mathbb{C}^{\times}$. Furthermore, for any quasi local subset S of $\mathcal{E}(W)$ and for any subgroup Γ of $\mathbb{C}^{\times}$, there exists a $\mathcal{Y}_{\Gamma}$-closed quasi local subspace of $\mathcal{E}(W)$, which contains S and 1_W.*

Proof. Clearly, the linear span of U and 1_W is quasi local. With U being maximal we must have $1_W \in U$. Let $a(x), b(x) \in U$, $\alpha \in \Gamma$, $n \in \mathbb{Z}$. By Proposition 4.3, for any $c(x) \in U$, $a(x)_{(\alpha,n)}b(x)$ and $c(x)$ are quasi local. In particular, $a(x)_{(\alpha,n)}b(x)$ is quasi local with $a(x)$ and $b(x)$. Using Proposition 4.3 again, we see that $a(x)_{(\alpha,n)}b(x)$ is quasi local with itself. Thus $U + \mathbb{C}a(x)_{(\alpha,n)}b(x)$ is quasi local. Since U is maximal, we must have $U + \mathbb{C}a(x)_{(\alpha,n)}b(x) = U$, which implies that $a(x)_{(\alpha,n)}b(x) \in U$. This proves that U is $\mathcal{Y}_{\Gamma}$-closed.

For any quasi local subset S of $\mathcal{E}(W)$, the linear span of S is also quasi local. It follows from Zorn's lemma that there exists a maximal quasi local subspace U of $\mathcal{E}(W)$, which contains S. By the first assertion, U is $\mathcal{Y}_{\Gamma}$-closed and it contains 1_W. Thus U is a $\mathcal{Y}_{\Gamma}$-closed quasi local subspace which contains S and 1_W. $\square$

In view of Corollary 4.5, for any quasi local subset S of $\mathcal{E}(W)$ and for any subgroup Γ of $\mathbb{C}^{\times}$, the smallest $\mathcal{Y}_{\Gamma}$-closed quasi local subspace, containing S and 1_W, of $\mathcal{E}(W)$ exists, which is the intersection of all $\mathcal{Y}_{\Gamma}$-closed quasi local subspaces containing S and 1_W.

Definition 4.6. Let S be any quasi local subspace of $\mathcal{E}(W)$. For any subgroup Γ of $\mathbb{C}^{\times}$, we define $\langle S \rangle_{\Gamma}$ to be the smallest $\mathcal{Y}_{\Gamma}$-closed quasi local subspace, containing S and 1_W, of $\mathcal{E}(W)$.

We also have the following results similar to Corollary 4.5:

Corollary 4.7. *Let Γ be any subgroup of $\mathbb{C}^{\times}$. Every maximal Γ-local subspace U of $\mathcal{E}(W)$ contains 1_W and is $\mathcal{Y}_{\Gamma}$-closed. For any Γ-local subset S of $\mathcal{E}(W)$, $\langle S \rangle_{\Gamma}$ is Γ-local and it is the smallest $\mathcal{Y}_{\Gamma}$-closed Γ-local subspace containing S and 1_W of $\mathcal{E}(W)$.*

Proof. From the proof of Corollary 4.5, using Proposition 4.3 we immediately see that every maximal Γ-local subspace U of $\mathcal{E}(W)$ contains 1_W and is $\mathcal{Y}_{\Gamma}$-closed. For any Γ-local subset S of $\mathcal{E}(W)$, by Zorn Lemma S is contained in some maximal Γ-local subspace U, which has been proved to be $\mathcal{Y}_{\Gamma}$-closed. Then $\langle S \rangle_{\Gamma} \subset U$. Consequently, $\langle S \rangle_{\Gamma}$ must be Γ-local. The rest is clear. $\square$

The following is the second key result:

Proposition 4.8. *Let Γ be a subgroup of $\mathbb{C}^\times$ and let U be a $\mathcal{Y}_\Gamma$-closed quasi local subspace of $\mathcal{E}(W)$. For any $a(x), b(x), c(x) \in U$, let $f(x_1, x_2)$ be a nonzero polynomial such that*

$$f(x_1, x_2)a(x_1)b(x_2) = f(x_1, x_2)b(x_2)a(x_1). \tag{4.11}$$

Then for $\alpha, \beta \in \Gamma$,

$$(x_1 - \alpha\beta^{-1}x_2)^s[\mathcal{Y}_\alpha(a(x), x_1), \mathcal{Y}_\beta(b(x), x_2)] = 0, \tag{4.12}$$

where s is the order of the zero of $f(x_1, x_2)$ at $x_1 = \alpha\beta^{-1}x_2$.

Proof. Let $g(x, y), h(x, y)$ be nonzero polynomials such that

$$
\begin{aligned}
g(x_1, x_2)a(x_1)c(x_2) &= g(x_1, x_2)c(x_2)a(x_1), \\
h(x_1, x_2)b(x_1)c(x_2) &= h(x_1, x_2)c(x_2)b(x_1).
\end{aligned}
$$

Let $\alpha, \beta \in \Gamma$ and let $n \in \mathbb{Z}$ be arbitrarily fixed. Since $b(x)_{(\beta,m)}c(x) = 0$ for m sufficiently large (by Proposition 3.6), from Proposition 4.3, there exists a nonnegative integer k depending on n such that for *all* $m \geq n$,

$$f(x_1, \beta x)^k g(x_1, x)a(x_1)(b(x)_{(\beta,m)}c(x)) = f(x_1, \beta x)^k g(x_1, x)(b(x)_{(\beta,m)}c(x))a(x_1). \tag{4.13}$$

Let $w \in W$ be arbitrarily fixed and let l be a nonnegative integer such that $x^l a(x)w \in W[[x]]$. By Proposition 3.6 we have

$$
\begin{aligned}
&f(x_1 + \alpha x, \beta x)^k g(x_1 + \alpha x, x)(x_1 + \alpha x)^l \mathcal{Y}_\alpha(a(x), x_1)(b(x)_{(\beta,m)}c(x))w \\
=\ &f(x_1 + \alpha x, \beta x)^k g(x_1 + \alpha x, x)(x_1 + \alpha x)^l a(x_1 + \alpha x)(b(x)_{(\beta,m)}c(x))w \tag{4.14}
\end{aligned}
$$

for *all* $m \geq n$. That is, for *all* $i \geq 0$,

$$
\begin{aligned}
&\mathrm{Res}_{x_2} x_2^{n+i} f(x_1 + \alpha x, \beta x)^k g(x_1 + \alpha x, x)(x_1 + \alpha x)^l \left(\mathcal{Y}_\alpha(a(x), x_1)\mathcal{Y}_\beta(b(x), x_2)c(x)\right) w \\
=\ &\mathrm{Res}_{x_2} x_2^{n+i} f(x_1 + \alpha x, \beta x)^k g(x_1 + \alpha x, x)(x_1 + \alpha x)^l a(x_1 + \alpha x) \left(\mathcal{Y}_\beta(b(x), x_2)c(x)\right) w. \\
& \tag{4.15}
\end{aligned}
$$

Since $f(x_1 + \alpha x, x_2 + \beta x)$ is a polynomial in x_1, x_2, x, we immediately have that for *all* $j \geq 0$,

$$
\begin{aligned}
&\mathrm{Res}_{x_2} x_2^{n+j} f(x_1 + \alpha x, \beta x)^k g(x_1 + \alpha x, x)(x_1 + \alpha x)^l f(x_1 + \alpha x, x_2 + \beta x) \\
&\quad \cdot \left(\mathcal{Y}_\alpha(a(x), x_1)\mathcal{Y}_\beta(b(x), x_2)c(x)\right) w \\
=\ &\mathrm{Res}_{x_2} x_2^{n+j} f(x_1 + \alpha x, \beta x)^k g(x_1 + \alpha x, x)(x_1 + \alpha x)^l f(x_1 + \alpha x, x_2 + \beta x) \\
&\quad \cdot a(x_1 + \alpha x) \left(\mathcal{Y}_\beta(b(x), x_2)c(x)\right) w. \tag{4.16}
\end{aligned}
$$

Now, let r be a nonnegative integer such that $x^r b(x)w \in W[[x]]$. By Proposition 3.6 we have

$$h(x_2 + \beta x, x)(x_2 + \beta x)^r \left(\mathcal{Y}_\beta(b(x), x_2)c(x)\right) w = h(x_2 + \beta x, x)(x_2 + \beta x)^r b(x_2 + \beta x)c(x)w.$$

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Using this, from (4.16) we obtain

$$\operatorname{Res}_{x_2} x_2^n f(x_1 + \alpha x, \beta x)^k g(x_1 + \alpha x, x)(x_1 + \alpha x)^l h(x_2 + \beta x, x)(x_2 + \beta x)^r$$
$$\cdot f(x_1 + \alpha x, x_2 + \beta x) \left(\mathcal{Y}_\alpha(a(x), x_1) \mathcal{Y}_\beta(b(x), x_2) c(x) \right) w$$
$$= \operatorname{Res}_{x_2} x_2^n f(x_1 + \alpha x, \beta x)^k g(x_1 + \alpha x, x)(x_1 + \alpha x)^l h(x_2 + \beta x, x)(x_2 + \beta x)^r$$
$$\cdot f(x_1 + \alpha x, x_2 + \beta x) a(x_1 + \alpha x) b(x_2 + \beta x) c(x) w. \tag{4.17}$$

Noticing that

$$g(x_1 + \alpha x, x)(x_1 + \alpha x)^l a(x_1 + \alpha x) c(x) w \in W((x))[[x_1]],$$

we have

$$g(x_1 + \alpha x, x)(x_1 + \alpha x)^l f(x_1 + \alpha x, x_2 + \beta x) a(x_1 + \alpha x) b(x_2 + \beta x) c(x) w$$
$$= g(x_1 + \alpha x, x)(x_1 + \alpha x)^l f(x_1 + \alpha x, x_2 + \beta x) b(x_2 + \beta x) a(x_1 + \alpha x) c(x) w,$$

which lies in $W((x))((x_1))[[x_2, x_2^{-1}]]$, a vector space over $\mathbb{C}((x))((x_1))$. The expression on the left hand side of (4.17) also lies in $W((x))((x_1))[[x_2, x_2^{-1}]]$. In view of Lemma 2.2 we can cancel the factor $f(x_1 + \alpha x, \beta x)^k$, to obtain

$$\operatorname{Res}_{x_2} x_2^n g(x_1 + \alpha x, x)(x_1 + \alpha x)^l h(x_2 + \beta x, x)(x_2 + \beta x)^r$$
$$\cdot f(x_1 + \alpha x, x_2 + \beta x) \left(\mathcal{Y}_\alpha(a(x), x_1) \mathcal{Y}_\beta(b(x), x_2) c(x) \right) w$$
$$= \operatorname{Res}_{x_2} x_2^n g(x_1 + \alpha x, x)(x_1 + \alpha x)^l h(x_2 + \beta x, x)(x_2 + \beta x)^r$$
$$\cdot f(x_1 + \alpha x, x_2 + \beta x) a(x_1 + \alpha x) b(x_2 + \beta x) c(x) w. \tag{4.18}$$

Notice that only k here depends on n (all the other components do not depend on n) and n is arbitrarily fixed. Then we can drop $\operatorname{Res}_{x_2} x_2^n$ to get

$$g(x_1 + \alpha x, x)(x_1 + \alpha x)^l h(x_2 + \beta x, x)(x_2 + \beta x)^r$$
$$\cdot f(x_1 + \alpha x, x_2 + \beta x) \left(\mathcal{Y}_\alpha(a(x), x_1) \mathcal{Y}_\beta(b(x), x_2) c(x) \right) w$$
$$= g(x_1 + \alpha x, x)(x_1 + \alpha x)^l h(x_2 + \beta x, x)(x_2 + \beta x)^r$$
$$\cdot f(x_1 + \alpha x, x_2 + \beta x) a(x_1 + \alpha x) b(x_2 + \beta x) c(x) w. \tag{4.19}$$

Symmetrically, there are nonnegative integers l', r' such that

$$g(x_1 + \alpha x, x)(x_1 + \alpha x)^{l'} h(x_2 + \beta x, x)(x_2 + \beta x)^{r'}$$
$$\cdot f(x_1 + \alpha x, x_2 + \beta x) \left(\mathcal{Y}_\beta(b(x), x_2) \mathcal{Y}_\alpha(a(x), x_1) c(x) \right) w$$
$$= g(x_1 + \alpha x, x)(x_1 + \alpha x)^l h(x_2 + \beta x, x)(x_2 + \beta x)^r$$
$$\cdot f(x_1 + \alpha x, x_2 + \beta x) b(x_2 + \beta x) a(x_1 + \alpha x) c(x) w. \tag{4.20}$$

Since

$$f(x_1 + \alpha x, x_2 + \beta x) a(x_1 + \alpha x) b(x_2 + \beta x) = f(x_1 + \alpha x, x_2 + \beta x) b(x_2 + \beta x) a(x_1 + \alpha x),$$

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combining (4.19) with (4.20), then setting $l'' = l + l'$ and $r'' = r + r$ we get

$$
\begin{aligned}
& g(x_1 + \alpha x, x)(x_1 + \alpha x)^{l''} h(x_2 + \beta x, x)(x_2 + \beta x)^{r''} \\
& \quad \cdot f(x_1 + \alpha x, x_2 + \beta x)\,(\mathcal{Y}_\alpha(a(x), x_1)\mathcal{Y}_\beta(b(x), x_2)c(x))\,w \\
= {} & g(x_1 + \alpha x, x)(x_1 + \alpha x)^{l''} h(x_2 + \beta x, x)(x_2 + \beta x)^{r''} \\
& \quad \cdot f(x_1 + \alpha x, x_2 + \beta x)\,(\mathcal{Y}_\beta(b(x), x_2)\mathcal{Y}_\alpha(a(x), x_1)c(x))\,w.
\end{aligned}
\tag{4.21}
$$

Noting that

$$
\begin{aligned}
(\mathcal{Y}_\alpha(a(x), x_1)\mathcal{Y}_\beta(b(x), x_2)c(x))\,w &\in W((x))((x_1))((x_2)) \subset W((x))((x_1))[[x_2, x_2^{-1}]] \\
(\mathcal{Y}_\beta(b(x), x_2)\mathcal{Y}_\alpha(a(x), x_1)c(x))\,w &\in W((x))((x_2))((x_1)) \subset W((x))((x_1))[[x_2, x_2^{-1}]]
\end{aligned}
$$

and that $W((x))((x_1))[[x_2, x_2^{-1}]]$ is a vector space over $\mathbb{C}((x))((x_1))$, in view of Lemma 2.2 we can cancel the factor $g(x_1 + \alpha x, x)(x_1 + \alpha x)^{l''}$ and similarly we can cancel the factor $h(x_2 + \beta x, x)(x_2 + \beta x)^{r''}$. (But we cannot cancel the factor $f(x_1 + \alpha x, x_2 + \beta x)$.) Then we get

$$
\begin{aligned}
& f(x_1 + \alpha x, x_2 + \beta x)\,(\mathcal{Y}_\alpha(a(x), x_1)\mathcal{Y}_\beta(b(x), x_2)c(x))\,w \\
= {} & f(x_1 + \alpha x, x_2 + \beta x)\,(\mathcal{Y}_\beta(b(x), x_2)\mathcal{Y}_\alpha(a(x), x_1)c(x))\,w.
\end{aligned}
\tag{4.22}
$$

Noticing that f does not depend on w, we have

$$
\begin{aligned}
& f(x_1 + \alpha x, x_2 + \beta x)\mathcal{Y}_\alpha(a(x), x_1)\mathcal{Y}_\beta(b(x), x_2)c(x) \\
= {} & f(x_1 + \alpha x, x_2 + \beta x)\mathcal{Y}_\beta(b(x), x_2)\mathcal{Y}_\alpha(a(x), x_1)c(x).
\end{aligned}
\tag{4.23}
$$

Write

$$
f(x_1, x_2) = (x_1 - \alpha\beta^{-1}x_2)^s f_0(x_1, x_2)),
$$

where s is a nonnegative integer and $f_0(x_1, x_2) \in \mathbb{C}[x_1, x_2]$ with $f_0(\alpha\beta^{-1}x_2, x_2) \neq 0$. We have

$$
f(x_1 + \alpha x, x_2 + \beta x) = (x_1 - \alpha\beta^{-1}x_2)^s f_0(x_1 + \alpha x, x_2 + \beta x)
$$

with $f_0(\alpha x, \beta x) \neq 0$. In view of Lemma 2.2 we can cancel the factor $f_0(x_1 + \alpha x, x_2 + \beta x)$ from (4.23). After cancelation we get

$$
\begin{aligned}
& (x_1 - \alpha\beta^{-1}x_2)^s \mathcal{Y}_\alpha(a(x), x_1)\mathcal{Y}_\beta(b(x), x_2)c(x) \\
= {} & (x_1 - \alpha\beta^{-1}x_2)^s \mathcal{Y}_\beta(b(x), x_2)\mathcal{Y}_\alpha(a(x), x_1)c(x).
\end{aligned}
\tag{4.24}
$$

This completes the proof. $\square$

Now we are in a position to present our main and key result:

Theorem 4.9. *Let W be a vector space, Γ a subgroup of $\mathbb{C}^\times$ and U a $\mathcal{Y}_\Gamma$-closed quasi local subspace of $\mathcal{E}(W)$, containing 1_W, in the sense that*

$$
a(x)_{(\alpha,n)}b(x) \in U \quad \text{for } a(x), b(x) \in U,\ \alpha \in \Gamma,\ n \in \mathbb{Z}.
\tag{4.25}
$$

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Then

$$\mathcal{Y}_\alpha(1_W, x_0)a(x) = a(x), \tag{4.26}$$

$$\mathcal{Y}_\alpha(a(x), x_0)1_W = e^{\alpha^{-1}x_0 D} R_\alpha a(x) \quad \text{for } a(x) \in U, \tag{4.27}$$

and the following identity holds for any $a(x), b(x) \in U$, $\alpha, \beta \in \Gamma$:

$$
x_0^{-1}\delta\left(\frac{x_1 - \alpha\beta^{-1}x_2}{x_0}\right) \mathcal{Y}_\alpha(a(x), x_1)\mathcal{Y}_\beta(b(x), x_2)
$$
$$
- x_0^{-1}\delta\left(\frac{-\alpha\beta^{-1}x_2 + x_1}{x_0}\right) \mathcal{Y}_\beta(b(x), x_2)\mathcal{Y}_\alpha(a(x), x_1)
$$
$$
= x_1^{-1}\delta\left(\frac{\alpha\beta^{-1}x_2 + x_0}{x_1}\right) \mathcal{Y}_\beta(\mathcal{Y}_{\alpha\beta^{-1}}(a(x), x_0)b(x), x_2). \tag{4.28}
$$

In particular, $(U, \mathcal{Y}_1, 1_W)$ *carries the structure of a vertex algebra. Furthermore, for any quasi local subset* S *of* $\mathcal{E}(W)$, $(\langle S\rangle_\Gamma, \mathcal{Y}_1, 1_W)$ *carries the structure of a vertex algebra, where* $\langle S\rangle_\Gamma$ *is the smallest* $\mathcal{Y}_\Gamma$-*closed quasi local subspace containing* S *and* 1_W.

Proof. It is clear that all the assertions follow if the Jacobi-like identity (4.28) is proved. With Propositions 3.8 and 3.9 and 4.8, it follows from [Li1] ([LL], cf. [DL], [FHL]) that $(U, \mathcal{Y}_1, 1_W)$ carries the structure of a vertex algebra. Thus (4.28) holds for $\alpha = \beta = 1$. The general case follows from this special case and the relation (3.31): For $u, v \in U$, $\alpha, \beta \in \Gamma$, we have

$$
x_0^{-1}\delta\left(\frac{\alpha^{-1}x_1 - \beta^{-1}x_2}{x_0}\right) \mathcal{Y}_1(R_\alpha u, \alpha^{-1}x_1)\mathcal{Y}_1(R_\beta v, \beta^{-1}x_2)
$$
$$
- x_0^{-1}\delta\left(\frac{-\beta^{-1}x_2 + \alpha^{-1}x_1}{x_0}\right) \mathcal{Y}_1(R_\beta v, \beta^{-1}x_2)\mathcal{Y}_1(R_\alpha u, \alpha^{-1}x_1)
$$
$$
= \alpha x_1^{-1}\delta\left(\frac{\beta^{-1}x_2 + x_0}{\alpha^{-1}x_1}\right) \mathcal{Y}_1(\mathcal{Y}_1(R_\alpha u, x_0)R_\beta v, \beta^{-1}x_2). \tag{4.29}
$$

By rewriting the three delta functions we get

$$
(\alpha x_0)^{-1}\delta\left(\frac{x_1 - \alpha\beta^{-1}x_2}{\alpha x_0}\right) \mathcal{Y}_1(R_\alpha u, \alpha^{-1}x_1)\mathcal{Y}_1(R_\beta v, \beta^{-1}x_2)
$$
$$
- (\alpha x_0)^{-1}\delta\left(\frac{-\alpha\beta^{-1}x_2 + x_1}{\alpha x_0}\right) \mathcal{Y}_1(R_\beta v, \beta^{-1}x_2)\mathcal{Y}_1(R_\alpha u, \alpha^{-1}x_1)
$$
$$
= x_1^{-1}\delta\left(\frac{\alpha\beta^{-1}x_2 + \alpha x_0}{x_1}\right) \mathcal{Y}_1(\mathcal{Y}_1(R_\alpha u, x_0)R_\beta v, \beta^{-1}x_2). \tag{4.30}
$$

In view of (3.31), we have

$$\mathcal{Y}_1(\mathcal{Y}_1(R_\alpha u, x_0)R_\beta v, \beta^{-1}x_2) = \mathcal{Y}_\beta(\mathcal{Y}_{\beta^{-1}\alpha}(u, \alpha x_0)v, x_2).$$

Replacing $x_0 \to \alpha^{-1}x_0$ we get the desired Jacobi identity $\qquad\qquad\square$

Similar to the situation for vertex subalgebras generated by subsets (cf. [LL]) we have:

Proposition 4.10. *Let S be a quasi local subset of $\mathcal{E}(W)$. Then for any subgroup Γ of $\mathbb{C}^\times$, $\langle S \rangle_\Gamma$ is linearly spanned by the vectors*

$$a^{(1)}(x)_{(\alpha_1, n_1)} \cdots a^{(r)}(x)_{(\alpha_r, n_r)} 1_W \tag{4.31}$$

for $r \geq 0$, $a^{(i)}(x) \in S$, $\alpha_i \in \Gamma$, $n_i \in \mathbb{Z}$.

Proof. Let J be the subspace of $\mathcal{E}(W)$, linearly spanned by those vectors in (4.31). It follows from the Jacobi-like identity (4.28) and induction that J is $\mathcal{Y}_\Gamma$-closed. Clearly, J contains S and 1_W. Thus, $\langle S \rangle_\Gamma \subset J$. On the other hand, any $\mathcal{Y}_\Gamma$-closed quasi local subspace containing S and 1_W must contain J, so that $J \subset \langle S \rangle_\Gamma$. This proves $\langle S \rangle_\Gamma = J$, completing the proof. $\square$

Definition 4.11. Let S be a quasi local subset of $\mathcal{E}(W)$. We set

$$\langle S \rangle = \langle S \rangle_{\{1\}}, \tag{4.32}$$

the smallest $\mathcal{Y}_1$-closed quasi local subspace, containing S and 1_W, of $\mathcal{E}(W)$. We call $\langle S \rangle$ the *vertex algebra generated by S*.

From definition we have $\langle S \rangle_\Gamma \subset \langle S \rangle_{\Gamma'}$ if $\Gamma \subset \Gamma'$ are subgroups of $\mathbb{C}^\times$. Then $\langle S \rangle$ is a vertex subalgebra of $\langle S \rangle_\Gamma$ for any subgroup Γ of $\mathbb{C}^\times$ and it is the vertex subalgebra of $\langle S \rangle_\Gamma$ generated by S.

Lemma 4.12. *Let W be a vector space, Γ a subgroup of $\mathbb{C}^\times$ and U a Γ-local subspace of $\mathcal{E}(W)$. If U is also a Γ-submodule of $\mathcal{E}(W)$, then $\langle U \rangle = \langle U \rangle_\Gamma$.*

Proof. With $\langle U \rangle = \langle U \rangle_{\{1\}} \subset \langle U \rangle_\Gamma$ we must prove $\langle U \rangle_\Gamma \subset \langle U \rangle$. Recall from (3.29) (Proposition 3.9) that

$$R_\alpha \mathcal{Y}_1(a(x), x_0) = \mathcal{Y}_1(R_\alpha a(x), \alpha^{-1} x_0) R_{\alpha^{-1}}.$$

Since U is a Γ-submodule of $\mathcal{E}(W)$, it follows from Proposition 4.10 (and induction) that $\langle U \rangle$ is a Γ-submodule of $\mathcal{E}(W)$. From (3.30) (Proposition 3.9) we have

$$\mathcal{Y}_\alpha(a(x), x_0) = \mathcal{Y}_1(R_\alpha a(x), \alpha^{-1} x_0)$$

It follows that $\langle U \rangle$ is $\mathcal{Y}_\Gamma$-closed. Then we have $\langle U \rangle_\Gamma \subset \langle U \rangle$, completing the proof. $\square$

5 Vertex algebras and their quasi modules

In this section we formulate and study the notion of quasi module for a vertex algebra. We strengthen some of the assertions of Theorem 4.9, showing that for any vector space W and for any $\mathcal{Y}_\Gamma$-closed quasi local subspace U of $\mathcal{E}(W)$ containing 1_W, W is a quasi

module for the vertex algebra $(U, \mathcal{Y}_1, 1_W)$. We also prove certain results analogous to those for modules.

First we study the possibility of a certain natural generalization of the notion of vertex algebra. It has been well known that the locality (namely weak commutativity) property plays a very important role in the theory of vertex algebras. With the notion of quasi-locality, one naturally considers to generalize the notion of vertex algebra by replacing the usual locality with quasi locality. The following proposition says that quasi locality does not give rise to something more general than vertex algebras.

Proposition 5.1. *Let V be a vector space equipped with a linear map Y from V to $\mathrm{Hom}\,(V, V((x)))$, a distinguished vector $\mathbf{1}$ and a linear operator $\mathcal{D}$ on V such that all the following conditions hold:*

$$\mathcal{D}(\mathbf{1}) = 0, \tag{5.1}$$

$$Y(\mathbf{1}, x) = 1, \tag{5.2}$$

$$Y(v, x)\mathbf{1} \in V[[x]] \quad and \quad \lim_{x \to 0} Y(v, x)\mathbf{1} = v, \tag{5.3}$$

$$[\mathcal{D}, Y(v, x)] = \frac{d}{dx}Y(v, x) \tag{5.4}$$

for $v \in V$ and such that for $u, v \in V$, there exists a nonzero polynomial $f(x_1, x_2)$ such that

$$f(x_1, x_2)Y(u, x_1)Y(v, x_2) = f(x_1, x_2)Y(v, x_2)Y(u, x_1). \tag{5.5}$$

Then $(V, Y, \mathbf{1})$ is an (ordinary) vertex algebra.

Proof. First, (5.4) together with the Taylor theorem immediately gives the following conjugation formula

$$e^{x\mathcal{D}}Y(v, x_0)e^{-x\mathcal{D}} = Y(v, x_0 + x) \quad \text{for } v \in V. \tag{5.6}$$

Applying this to $\mathbf{1}$, using (5.1) and (5.3) (by setting $x_0 = 0$) we get

$$Y(v, x)\mathbf{1} = e^{x\mathcal{D}}v \quad \text{for } v \in V.$$

Let $u, v, w \in V$ and let $f(x_1, x_2)$ be a nonzero polynomial such that (5.5) holds. We have

$$
\begin{aligned}
Y(Y(u, x_1)Y(v, x_2)w, x)\mathbf{1} &= e^{x\mathcal{D}}Y(u, x_1)Y(v, x_2)w \\
&= Y(u, x_1 + x)Y(v, x_2 + x)e^{x\mathcal{D}}w, \tag{5.7} \\
Y(Y(v, x_2)Y(u, x_1)w, x)\mathbf{1} &= Y(v, x_2 + x)Y(u, x_1 + x)e^{x\mathcal{D}}w. \tag{5.8}
\end{aligned}
$$

Using these relations and (5.5) we get

$$
\begin{aligned}
&f(x_1 + x, x_2 + x)Y(Y(u, x_1)Y(v, x_2)w, x)\mathbf{1} \\
&= f(x_1 + x, x_2 + x)Y(Y(v, x_2)Y(u, x_1)w, x)\mathbf{1}. \tag{5.9}
\end{aligned}
$$

Factor $f(x_1, x_2)$ as

$$f(x_1, x_2) = (x_1 - x_2)^k f_0(x_1, x_2),$$

where $k \geq 0$, $f_0(x_1, x_2) \in \mathbb{C}[x_1, x_2]$ with $f_0(x_1, x_1) \neq 0$. Then we have

$$
\begin{aligned}
&(x_1 - x_2)^k f_0(x_1 + x, x_2 + x) Y(Y(u, x_1) Y(v, x_2) w, x) \mathbf{1} \\
= \ &(x_1 - x_2)^k f_0(x_1 + x, x_2 + x) Y(Y(v, x_2) Y(u, x_1) w, x) \mathbf{1}.
\end{aligned}
\tag{5.10}
$$

Notice that

$$
\begin{aligned}
Y(Y(u, x_1) Y(v, x_2) w, x) \mathbf{1} &\in V[[x]]((x_1))((x_2)), \\
Y(Y(v, x_2) Y(u, x_1) w, x) \mathbf{1} &\in V[[x]]((x_2))((x_1)).
\end{aligned}
$$

In view of Lemma 2.2, from (5.10) we get

$$(x_1 - x_2)^k Y(Y(u, x_1) Y(v, x_2) w, x) \mathbf{1} = (x_1 - x_2)^k Y(Y(v, x_2) Y(u, x_1) w, x) \mathbf{1}. \tag{5.11}$$

In view of (5.3), setting $x = 0$ we obtain

$$(x_1 - x_2)^k Y(u, x_1) Y(v, x_2) w = (x_1 - x_2)^k Y(v, x_2) Y(u, x_1) w. \tag{5.12}$$

This proves that the usual locality (weak commutativity) holds. Then it follows from [Li1] ([LL], cf. [DL], [FHL]) that V is a vertex algebra. $\qquad\square$

Even though the notion of quasi locality does not lead to a generalization of the notion of vertex algebra, it does lead to a new notion of what we call quasi module for an (ordinary) vertex algebra.

Definition 5.2. Let V be a vertex algebra. A *quasi V-module* is a vector space W equipped with a linear map Y_W from V to $\mathrm{Hom}\,(W, W((x)))$ such that

$$Y_W(\mathbf{1}, x) = 1_W \tag{5.13}$$

and such that for any $u, v \in V$, there exists a nonzero polynomial $f(x_1, x_2)$ such that

$$
\begin{aligned}
&x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) f(x_1, x_2) Y_W(u, x_1) Y_W(v, x_2) \\
&- x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) f(x_1, x_2) Y_W(v, x_2) Y_W(u, x_1) \\
= \ &x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) f(x_1, x_2) Y_W(Y(u, x_0) v, x_2).
\end{aligned}
\tag{5.14}
$$

For convenience we refer the identity (5.14) as the *quasi Jacobi identity for (u, v)*.

Lemma 5.3. *Let (W, Y_W) be a quasi module for a vertex algebra V. Then*

$$Y_W(\mathcal{D}v, x) = \frac{d}{dx} Y_W(v, x) \quad \text{for } v \in V. \tag{5.15}$$

Proof. It is similar to the proof for usual modules (cf. [LL]). Let $v \in V$. There exists $0 \neq f(x_1, x_2) \in \mathbb{C}[x_1, x_2]$ such that the quasi Jacobi identity for $(v, \mathbf{1})$ holds. Taking Res_{x_1} and using $Y_W(\mathbf{1}, x) = 1_W$, we get

$$
\begin{aligned}
& f(x_2 + x_0, x_2) Y_W(Y(v, x_0)\mathbf{1}, x_2) \\
= \ & \mathrm{Res}_{x_1}\left(x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) f(x_1, x_2) Y_W(v, x_1) - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right) f(x_1, x_2) Y_W(v, x_1)\right) \\
= \ & \mathrm{Res}_{x_1} x_1^{-1}\delta\left(\frac{x_2 + x_0}{x_1}\right) f(x_1, x_2) Y_W(v, x_1) \\
= \ & f(x_2 + x_0, x_2) Y_W(v, x_2 + x_0).
\end{aligned}
\tag{5.16}
$$

Noticing that both $Y_W(Y(v, x_0)\mathbf{1}, x_2)$ and $Y_W(v, x_2 + x_0)$ involve only nonnegative powers of x_0, we can apply the cancelation law (Lemma 2.2), to get

$$
Y_W(Y(v, x_0)\mathbf{1}, x_2) = Y_W(v, x_2 + x_0) = e^{x_0 \frac{\partial}{\partial x_2}} Y_W(v, x_2).
$$

From this we immediately have (5.15). $\qquad\square$

With the notion of quasi module we have the following important result:

Theorem 5.4. *Let W be any vector space, let Γ be a subgroup of $\mathbb{C}^\times$ and let U be a $\mathcal{Y}_\Gamma$-closed quasi local subspace of $\mathcal{E}(W)$, containing 1_W. Then (W, Y_W) carries the structure of a quasi module for the vertex algebra $(U, \mathcal{Y}_1, 1_W)$ obtained in Theorem 4.9, where*

$$
Y_W(a(x), x_0) = a(x_0) \quad \text{for } a(x) \in U.
\tag{5.17}
$$

Furthermore, for any quasi local subset S of $\mathcal{E}(W)$, (W, Y_W) carries the structure of a quasi module for the vertex algebra $\langle S \rangle_\Gamma$.

Proof. Clearly, we only need to prove the first assertion. For $a(x), b(x) \in U$, let $f(x_1, x_2)$ be a nonzero polynomial such that

$$
f(x_1, x_2) a(x_1) b(x_2) = f(x_1, x_2) b(x_2) a(x_1).
\tag{5.18}
$$

We have

$$
\begin{aligned}
f(x_1, x_2) Y_W(a(x), x_1) Y_W(b(x), x_2) &= f(x_1, x_2) a(x_1) b(x_2) \\
&= f(x_1, x_2) b(x_2) a(x_1) \\
&= f(x_1, x_2) Y_W(b(x), x_2) Y_W(a(x), x_1). \quad (5.19)
\end{aligned}
$$

From (5.18) we have that $f(x_1, x_2) a(x_1) b(x_2) \in \mathrm{Hom}\,(W, W((x_1, x_2)))$. Let $w \in W$ and let l be a nonnegative integer such that $x^l a(x) w \in W[[x]]$, so that

$$
x_1^l f(x_1, x_2) a(x_1) b(x_2) w \in W[[x_1, x_2]][x_2^{-1}].
$$

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With

$$Y_W(a(x), x_0 + x_2)Y_W(b(x), x_2)w = a(x_0 + x_2)b(x_2)w,$$
$$Y_W(\mathcal{Y}_1(a(x), x_0)b(x), x_2)w = (\mathcal{Y}_1(a(x), x_0)b(x)) w|_{x=x_2},$$

from Proposition 3.6 we have

$$
(x_0 + x_2)^l f(x_0 + x_2, x_2)Y_W(a(x), x_0 + x_2)Y_W(b(x), x_2)w
$$
$$
= (x_0 + x_2)^l f(x_0 + x_2, x_2)Y_W(\mathcal{Y}_1(a(x), x_0)b(x), x_2)w. \tag{5.20}
$$

In view of Lemma 2.4, the required quasi Jacobi identity for (u, v) holds. By definition $Y_W(1_W, x_0) = 1_W$. This proves that W is a quasi module. $\square$

The following result says that for a vertex algebra V, the notion of quasi V-module is canonically equivalent to a notion of "quasi representation":

Theorem 5.5. *Let V be a vertex algebra and let W be a vector space equipped with a linear map Y_W from V to $\mathcal{E}(W)$. Then (W, Y_W) carries the structure of a quasi V-module if and only if $\{Y_W(v, x) \mid v \in V\}$ is a $\mathcal{Y}_1$-closed quasi local subspace containing 1_W of $\mathcal{E}(W)$ and Y_W is a vertex algebra homomorphism.*

Proof. Assume that (W, Y_W) is a quasi V-module. Set

$$U = \{Y_W(v, x) \mid v \in V\} \subset \mathcal{E}(W).$$

For any $u, v \in V$, there exists $0 \neq f(x_1, x_2) \in \mathbb{C}[x_1, x_2]$ such that

$$
x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) f(x_1, x_2)Y_W(u, x_1)Y_W(v, x_2)
$$
$$
-x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right) f(x_1, x_2)Y_W(v, x_2)Y_W(u, x_1)
$$
$$
= x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) f(x_1, x_2)Y_W(Y(u, x_0)v, x_2), \tag{5.21}
$$

which implies

$$(x_1 - x_2)^k f(x_1, x_2)[Y_W(u, x_1), Y_W(v, x_2)] = 0 \tag{5.22}$$

for some nonnegative integer k with $x^k Y(u, x)v \in V[[x]]$. This shows that U is a quasi local subspace of $\mathcal{E}(W)$. Furthermore, with (5.22), using Proposition 3.10 we get

$$
x_1^{-1}\delta\left(\frac{x_2 + x_0}{x_1}\right) f(x_1, x_2)(x_1 - x_2)^k \mathcal{Y}_1(Y_W(u, x_2), x_0)Y_W(v, x_2)
$$
$$
= x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) f(x_1, x_2)(x_1 - x_2)^k Y_W(u, x_1)Y_W(v, x_2)
$$
$$
-x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right) f(x_1, x_2)(x_1 - x_2)^k Y_W(v, x_2)Y_W(u, x_1)
$$
$$
= x_1^{-1}\delta\left(\frac{x_2 + x_0}{x_1}\right) f(x_1, x_2)(x_1 - x_2)^k Y_W(Y(u, x_0)v, x_2). \tag{5.23}
$$

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Taking Res_{x_1} we get

$$f(x_2 + x_0, x_2)x_0^k \mathcal{Y}_1(Y_W(u, x_2), x_0)Y_W(v, x_2) = f(x_2 + x_0, x_2)x_0^k Y_W(Y(u, x_0)v, x_2). \quad (5.24)$$

Since both $\mathcal{Y}_1(Y_W(u, x_2), x_0)Y_W(v, x_2)$ and $Y_W(Y(u, x_0)v, x_2)$ involve only finitely many negative powers of x_0, using the cancelation law (Lemma 2.2) we get

$$\mathcal{Y}_1(Y_W(u, x_2), x_0)Y_W(v, x_2) = Y_W(Y(u, x_0)v, x_2). \quad (5.25)$$

This implies that U is $\mathcal{Y}_1$-closed. The space U contains 1_W since $Y_W(\mathbf{1}, x) = 1_W$. From (5.25), $(U, \mathcal{Y}_1, 1_W)$ is a vertex algebra and the linear map Y_W is a vertex algebra homomorphism.

Conversely, assume that Y_W is a (vertex algebra) homomorphism from V to a $\mathcal{Y}_1$-closed quasi local subspace containing 1_W of $\mathcal{E}(W)$. We have $Y_W(\mathbf{1}, x) = 1_W$. For $u, v \in V$, let $0 \neq f(x_1, x_2) \in \mathbb{C}[x_1, x_2]$ such that

$$f(x_1, x_2)Y_W(u, x_1)Y_W(v, x_2) = f(x_1, x_2)Y_W(v, x_2)Y_W(u, x_1).$$

By Proposition 3.10 we have

$$
\begin{aligned}
& x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) f(x_1, x_2)Y_W(u, x_1)Y_W(v, x_2) \\
& -x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right) f(x_1, x_2)Y_W(v, x_2)Y_W(u, x_1) \\
= \; & x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) f(x_1, x_2)Y_1(Y_W(u, x_2), x_0)Y_W(v, x_2) \\
= \; & x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) f(x_1, x_2)Y_W(Y(u, x_0)v, x_2). \quad (5.26)
\end{aligned}
$$

Thus (W, Y_W) is a quasi V-module. $\quad\square$

The following is an analogue of Theorem 5.7.6 of [LL]:

Theorem 5.6. *Let V be a vertex algebra with a generating subspace U and let W be a vector space equipped with a linear map Y_W^0 from U to $\text{Hom}(W, W((x)))$. Then Y_W^0 can be extended to a linear map Y_W from V to $\text{Hom}(W, W((x)))$ such that (W, Y_W) carries the structure of a quasi V-module if and only if $\bar{U} = \{Y_W^0(u, x) \mid u \in U\}$ is a quasi local subspace of $\mathcal{E}(W)$ and there exists a linear map ψ from V to $\langle\bar{U}\rangle$ such that*

$$\psi(\mathbf{1}) = 1_W, \quad (5.27)$$
$$\psi(Y(u, x_0)v) = \mathcal{Y}_1(Y_W^0(u, x), x_0)\psi(v) \quad (5.28)$$

for $u \in U$, $v \in V$. Furthermore, such an extension is unique.

Proof. If Y_W is an extension of Y_W^0 such that (W, Y_W) carries the structure of a quasi V-module, then by Theorem 5.5, $\{Y_W(v, x) \mid v \in V\}$ is a $\mathcal{Y}_1$-closed quasi local subspace of $\mathcal{E}(W)$ and Y_W is a vertex algebra homomorphism. Since V is generated by U, $Y_W(V)$ is generated by $\bar{U}$, i.e., $Y_W(V) = \langle \bar{U} \rangle$. Taking $\psi = Y_W$ we have (5.27) and (5.28).

On the other hand, assume that $\bar{U}$ is a quasi local subspace of $\mathcal{E}(W)$ and ψ is a linear map from V to $\langle \bar{U} \rangle$ satisfying (5.27) and (5.28). Since V and $\langle \bar{U} \rangle$ are vertex algebras and U generates V, by Proposition 5.7.9 of [LL] ψ is a vertex algebra homomorphism. In view of Theorem 5.5, (W, Y_W) carries the structure of a quasi V-module with $Y_W(v, x) = \psi_x(v)$ for $v \in V$. For $u \in U$, using (5.27) and (5.28) we have

$$Y_W(u, x) = \psi_x(u) = \psi_x(u_{-1}\mathbf{1}) = Y_W^0(u, x)_{-1}1_W = Y_W^0(u, x). \tag{5.29}$$

Thus Y_W extends Y_W^0.

Since U generates V, any linear map from U to a vertex algebra V' extends to at most one vertex algebra homomorphism from V to V'. Then the uniqueness follows. $\square$

The following is an analogue of a result in [Li1] (cf. [LL], Theorem 3.6.3):

Proposition 5.7. *Let V be a vertex algebra and let W be a vector space equipped with a linear map Y_W from V to $\mathrm{Hom}\,(W, W((x)))$ such that $Y_W(\mathbf{1}, x) = 1_W$ and for $u, v \in V$ there exists a nonzero polynomial $f(x_1, x_2)$ such that for $w \in W$*

$$f(x_0 + x_2, x_2)(x_0 + x_2)^l Y_W(u, x_0 + x_2) Y_W(v, x_2) w$$
$$= f(x_0 + x_2, x_2)(x_0 + x_2)^l Y_W(Y(u, x_0)v, x_2) w \tag{5.30}$$

for some nonnegative integer l. Then (W, Y_W) carries the structure of a quasi V-module.

Proof. First we prove that

$$Y_W(\mathcal{D}(u), x) = \frac{d}{dx} Y_W(u, x) \quad \text{for } u \in V. \tag{5.31}$$

Let $u \in V$, $w \in W$. Then there exist a nonzero polynomial $f(x_1, x_2)$ and a nonnegative integer l such that

$$f(x_0 + x_2, x_2)(x_0 + x_2)^l Y_W(u, x_0 + x_2) Y_W(\mathbf{1}, x_2) w$$
$$= f(x_0 + x_2, x_2)(x_0 + x_2)^l Y_W(Y(u, x_0)\mathbf{1}, x_2) w.$$

With $Y_W(\mathbf{1}, x) = 1_W$ we have

$$f(x_0 + x_2, x_2)(x_0 + x_2)^l Y_W(u, x_0 + x_2) w = f(x_0 + x_2, x_2)(x_0 + x_2)^l Y_W(Y(u, x_0)\mathbf{1}, x_2) w.$$

We may replace l with a bigger integer if necessary so that $x^l Y_W(u, x) w \in W[[x]]$. Then

$$(x_0 + x_2)^l Y_W(u, x_0 + x_2) w = (x_0 + x_2)^l Y_W(u, x_2 + x_0) w.$$

We thus have

$$f(x_0 + x_2, x_2)(x_0 + x_2)^l Y_W(u, x_2 + x_0)w = f(x_0 + x_2, x_2)(x_0 + x_2)^l Y_W(Y(u, x_0)\mathbf{1}, x_2)w.$$

In view of the cancelation law (Lemma 2.2) we have

$$Y_W(u, x_2 + x_0)w = Y_W(Y(u, x_0)\mathbf{1}, x_2)w,$$

which implies $\frac{d}{dx}Y_W(u, x)w = Y_W(\mathcal{D}(u), x)w$. This proves (5.31).

Let $u, v \in V$. Using the skew symmetry of V and (5.31) we have

$$Y_W(Y(u, x_0)v, x_2) = Y_W(e^{x_0 \mathcal{D}}Y(v, -x_0)u, x_2) = Y_W(Y(v, -x_0)u, x_2 + x_0). \qquad (5.32)$$

Let $w \in W$. There exist nonzero polynomials $f(x_1, x_2)$ and $g(x_1, x_2)$ depending only on u, v and nonnegative integers l, l' such that

$$\begin{aligned}
&(x_0 + x_2)^l f(x_0 + x_2, x_2)Y_W(u, x_0 + x_2)Y_W(v, x_2)w \\
= &(x_0 + x_2)^l f(x_0 + x_2, x_2)Y_W(Y(u, x_0)v, x_2)w, \qquad (5.33) \\
&(-x_0 + x_1)^{l'} g(x_1, -x_0 + x_1)Y_W(v, -x_0 + x_1)Y_W(u, x_1)w \\
= &(-x_0 + x_1)^{l'} g(x_1, -x_0 + x_1)Y_W(Y(v, -x_0)u, x_1)w. \qquad (5.34)
\end{aligned}$$

Replacing l with a larger one if necessary so that $x^l Y_W(u, x)w \in W[[x]]$. Using (5.34) we see that the expression

$$x_1^l(-x_0 + x_1)^{l'} g(x_1, -x_0 + x_1)Y_W(Y(v, -x_0)u, x_1)w$$

lies in $W[[x_0, x_1]][x_0^{-1}]$. In view of the delta function substitution rule (2.14) we have

$$\begin{aligned}
&x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right)\left(x_1^l f(x_1, x_2)(-x_0 + x_1)^{l'} g(x_1, -x_0 + x_1)Y_W(Y(v, -x_0)u, x_1)w\right) \\
= &x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right)\left(x_1^l f(x_1, x_2)(-x_0 + x_1)^{l'} g(x_1, -x_0 + x_1)Y_W(Y(v, -x_0)u, x_1)w\right)\big|_{x_1 = x_2 + x_0} \\
= &x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right)\left((x_2 + x_0)^l f(x_2 + x_0, x_2)x_2^{l'} g(x_1, x_2)Y_W(Y(v, -x_0)u, x_2 + x_0)w\right).
\end{aligned}$$

$$(5.35)$$

Using (5.33), (5.34), (5.32) and (5.35) in a sequence we get

$$x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)x_1^l f(x_1,x_2)x_2^{l'}g(x_1,x_2)Y_W(u,x_1)Y_W(v,x_2)w$$

$$-x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)x_1^l f(x_1,x_2)x_2^{l'}g(x_1,x_2)Y_W(v,x_2)Y_W(u,x_1)w$$

$$= x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)x_2^{l'}g(x_1,x_2)(x_0+x_2)^l f(x_0+x_2,x_2)Y_W(u,x_0+x_2)Y_W(v,x_2)w$$

$$-x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)x_1^l f(x_1,x_2)(-x_0+x_1)^{l'}g(x_1,-x_0+x_1)Y_W(v,-x_0+x_1)Y_W(u,x_1)w$$

$$= x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)\left(x_2^{l'}g(x_1,x_2)(x_0+x_2)^l f(x_0+x_2,x_2)Y_W(Y(u,x_0)v,x_2)w\right)$$

$$-x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)\left(x_1^l f(x_1,x_2)(-x_0+x_1)^{l'}g(x_1,-x_0+x_1)Y_W(Y(v,-x_0)u,x_1)w\right)$$

$$= x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)\left(x_2^{l'}g(x_1,x_2)(x_0+x_2)^l f(x_0+x_2,x_2)Y_W(Y(v,-x_0)u,x_2+x_0)w\right)$$

$$-x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)\left(x_1^l f(x_1,x_2)(x_0+x_2)^{l'}g(x_1,-x_0+x_1)Y_W(Y(v,-x_0)u,x_1)w\right)$$

$$= x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)\left(x_2^{l'}g(x_1,x_2)(x_0+x_2)^l f(x_0+x_2,x_2)Y_W(Y(v,-x_0)u,x_2+x_0)w\right)$$

$$-x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)\left(x_2^{l'}g(x_1,x_2)(x_0+x_2)^l f(x_0+x_2,x_2)Y_W(Y(v,-x_0)u,x_2+x_0)w\right)$$

$$= x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)x_1^l f(x_1,x_2)x_2^{l'}g(x_1,x_2)Y_W(Y(u,x_0)v,x_2)w. \tag{5.36}$$

Multiplying by $x_1^{-l}x_2^{-l'}$ we obtain the quasi Jacobi identity for (u,v). This completes the proof. $\qquad\square$

The following result is an analogue of a theorem of [LL] (see also [R]):

Theorem 5.8. *Let W be a vector space and let U be a subspace of $\mathcal{E}(W)$ in which every pair is compatible. Assume that U is $\mathcal{Y}_1$-closed and it contains 1_W. Then $(U,\mathcal{Y}_1,1_W)$ is a vertex algebra if and only if U is quasi local.*

Proof. In view of Theorem 4.9 we only need to prove that if $(U,\mathcal{Y}_1,1_W)$ is a vertex algebra then U is quasi local. For $a(x),b(x)\in U$, let $f(x_1,x_2)$ be a nonzero polynomial such that

$$f(x_1,x_2)a(x_1)b(x_2)\in \mathrm{Hom}\,(W,W((x_1,x_2))).$$

For $w\in W$, let l be a nonnegative integer such that

$$x_1^l f(x_1,x_2)a(x_1)b(x_2)w\in W[[x_1,x_2]][x_2^{-1}].$$

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By Proposition 3.6 we have

$$f(x_0 + x_2, x_2)(x_0 + x_2)^l \left(\mathcal{Y}(a(x_2), x_0) b(x_2) \right) w$$
$$= f(x_0 + x_2, x_2)(x_0 + x_2)^l a(x_0 + x_2) b(x_2) w. \tag{5.37}$$

In view of Proposition 5.7, W is a quasi U-module with $Y_W(a(x), x_0) = a(x_0)$ for $a(x) \in U$. It follows from the quasi Jacobi identity that any $a(x), b(x)$ in U are quasi local. This proves that U is quasi local. $\square$

An important problem could be the classification and construction of irreducible quasi modules for a vertex algebra. In the following we present certain results in this direction.

Lemma 5.9. *Let V be a vertex algebra and (W, Y_W) a quasi V-module. Let $a, b, c \in V$ and let*

$$0 \neq f(x_1, x_2), g(x_1, x_2), h(x_1, x_2) \in \mathbb{C}[x_1, x_2]$$

such that

$$f(x_1, x_2)[Y_W(a, x_1), Y_W(b, x_2)] = 0, \tag{5.38}$$
$$g(x_1, x_2)[Y_W(a, x_1), Y_W(c, x_2)] = 0, \tag{5.39}$$
$$h(x_1, x_2)[Y_W(b, x_1), Y_W(c, x_2)] = 0. \tag{5.40}$$

Then for any $n \in \mathbb{Z}$, there exists a nonnegative integer k such that

$$f(x_1, x_2)^k g(x_1, x_2)[Y_W(a, x_1), Y_W(b_n c, x_2)] = 0. \tag{5.41}$$

Proof. In view of Theorem 5.5, for any $u, v \in V$, $Y_W(u, x)$ and $Y_W(v, x)$ are quasi local and

$$Y_W(u_m v, x) = Y_W(u, x)_m Y_W(v, x) \quad \text{for } m \in \mathbb{Z}. \tag{5.42}$$

From Proposition 4.3 there exists a nonnegative integer k such that

$$f(x_1, x_2)^k g(x_1, x_2) [Y_W(a, x_1), Y_W(b, x_2)_n Y_W(c, x_2)] = 0. \tag{5.43}$$

Thus we obtain

$$f(x_1, x_2)^k g(x_1, x_2)[Y_W(a, x_1), Y_W(b_n c, x_2)] = 0, \tag{5.44}$$

completing the proof. $\square$

From Proposition 5.9 and induction we immediately have:

Proposition 5.10. *Let V be a vertex algebra and (W, Y_W) a quasi V-module. Let U be a generating subspace of V and let $0 \neq f(x_1, x_2) \in \mathbb{C}[x_1, x_2]$ such that for any $u, u' \in U$ there exists a nonnegative integer k such that*

$$f(x_1, x_2)^k [Y_W(u, x_1), Y_W(u', x_2)] = 0. \tag{5.45}$$

Then for any $v, v' \in V$, there exists a nonnegative integer r such that

$$f(x_1, x_2)^r [Y_W(v, x_1), Y_W(v', x_2)] = 0. \tag{5.46}$$

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As immediate consequences we have:

Corollary 5.11. *Let W be any vector space and let U be a quasi local subspace of $\mathcal{E}(W)$. Assume that $f(x_1, x_2)$ is a nonzero polynomial such that for any $a(x), b(x) \in U$ there exists a nonnegative integer k such that*

$$f(x_1, x_2)^k[a(x_1), b(x_2)] = 0. \tag{5.47}$$

Let $V = \langle U \rangle$ be the vertex algebra generated by U. Then for any $\alpha(x), \beta(x) \in V$, there exists a nonnegative integer r such that

$$f(x_1, x_2)^r[\alpha(x_1), \beta(x_2)] = 0. \tag{5.48}$$

Corollary 5.12. *Let V be a finitely generated vertex algebra and let (W, Y_W) be any quasi V-module. Then there exists a nonzero polynomial $f(x_1, x_2)$ such that for any $u, v \in V$,*

$$f(x_1, x_2)^r[Y_W(u, x_1), Y_W(v, x_2)] = 0 \tag{5.49}$$

for some nonnegative integer r.

In view of Corollary 5.12 we would like to associate a "minimal" polynomial $f(x_1, x_2)$ to a quasi module W for a vertex algebra V. Since $\mathbb{C}[x_1, x_2]$ is not a principal domain, we have a technical problem uniquely to define such a term. Now we consider a special case. Assume that there exists a nonzero homogeneous polynomial $f(x_1, x_2)$ such that the assertion of Corollary 5.12 holds. Notice that a nonzero polynomial $g(x_1, x_2)$ is homogeneous if and only if $g(x_1, x_2) = x_2^k p(x_1/x_2)$ for some nonzero polynomial $p(x)$ of degree k.

Definition 5.13. Let V be a vertex algebra and W a quasi V-module satisfying that there exists a nonzero polynomial $p(x)$ such that for any $u, v \in V$,

$$p(x_1/x_2)^r[Y_W(u, x_1), Y_W(v, x_2)] = 0 \tag{5.50}$$

for some nonnegative integer r. The monic polynomial $p(x)$ of the least degree is called the *minimal polynomial* of W.

6 Γ-vertex algebras and (quasi) modules

Motivated by Theorem 4.9 and by the notion of Γ-conformal algebra in [G-K-K], in this section we formulate and study a notion of Γ-vertex algebra, where Γ is an arbitrary group. We show that a Γ-vertex algebra amounts to an (ordinary) vertex algebra equipped with a compatible Γ-module structure in a certain sense. We also formulate a notion of quasi module for a Γ-vertex algebra. In terms of these notions, we strengthen Theorem 4.9, showing that for any vector space W, any subgroup Γ of $\mathbb{C}^\times$ and for any $\mathcal{Y}_\gamma$-closed quasi local subspace U of $\mathcal{E}(W)$ containing 1_W, $(U, \{\mathcal{Y}_\alpha\}, 1_W)$ carries the structure of a Γ-vertex algebra with W as a quasi module.

Let Γ be any (possibly nonabelian and infinite) group. We shall use 1 for the identity element of Γ. Let

$$\phi : \Gamma \to \mathbb{C}^{\times} \tag{6.1}$$

be a group homomorphism. We fix the pair (Γ, ϕ) throughout this section.

An important example for such a pair (Γ, ϕ) is the one with Γ a subgroup of $\mathbb{C}^{\times}$ and with ϕ the identity map. As a convention, whenever Γ is given as a subgroup of $\mathbb{C}^{\times}$ *we always assume that ϕ is the identity map.*

With the group homomorphism ϕ, any $\mathbb{C}$-vector space W naturally becomes a Γ-module by defining

$$\alpha w = \phi(\alpha)w \quad \text{for } \alpha \in \Gamma, \ w \in W. \tag{6.2}$$

In particular, for formal variable x, with $\mathbb{C}[[x, x^{-1}]]$ as a $\mathbb{C}$-vector space we have

$$\alpha x = \phi(\alpha)x \quad \text{for } \alpha \in \Gamma. \tag{6.3}$$

We have

$$\alpha \beta x = \beta \alpha x \quad \text{for } \alpha, \beta \in \Gamma, \tag{6.4}$$

since $\phi(\alpha\beta) = \phi(\alpha)\phi(\beta) = \phi(\beta)\phi(\alpha) = \phi(\beta\alpha)$. Due to our convention, whenever $\Gamma \subset \mathbb{C}^{\times}$, the new Γ-action through ϕ agrees with the old Γ-action by scalar multiplication.

The following notion naturally arises from Theorem 4.9:

Definition 6.1. A Γ-*vertex algebra* is a vector space V equipped with linear maps

$$\begin{aligned} Y_\alpha : V &\to \text{Hom}\,(V, V((x))) \subset (\text{End}\,V)[[x, x^{-1}]] \\ v &\mapsto Y_\alpha(v, x) \end{aligned} \tag{6.5}$$

for $\alpha \in \Gamma$ and equipped with a distinguished vector $\mathbf{1}$, called the *vacuum vector*, such that all the following axioms hold: For $\alpha \in \Gamma$, $v \in V$,

$$Y_\alpha(\mathbf{1}, x) = 1, \tag{6.6}$$

$$Y_\alpha(v, x)\mathbf{1} = V[[x]] \quad \text{and} \quad \lim_{x \to 0} Y_1(v, x)\mathbf{1} = v \tag{6.7}$$

and for $u, v, w \in V$, $\alpha, \beta \in \Gamma$,

$$\begin{aligned} x_0^{-1}\delta &\left(\frac{x_1 - \beta^{-1}\alpha x_2}{x_0} \right) Y_\alpha(u, x_1)Y_\beta(v, x_2)w \\ -x_0^{-1}\delta &\left(\frac{-\beta^{-1}\alpha x_2 + x_1}{x_0} \right) Y_\beta(v, x_2)Y_\alpha(u, x_1)w \\ = \ x_1^{-1}\delta &\left(\frac{\beta^{-1}\alpha x_2 + x_0}{x_1} \right) Y_\beta(Y_{\beta^{-1}\alpha}(u, x_0)v, x_2)w. \end{aligned} \tag{6.8}$$

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The Γ-vertex algebra V is also often denoted by $(V, \{Y_\alpha\}, \mathbf{1})$. For convenience, we refer (6.8) as the Γ-*Jacobi identity for the triple* (u, v, w).

Remark 6.2. It is clear that the notion of Γ-vertex algebra with $\Gamma = \{1\}$ reduces to the notion of ordinary vertex algebra.

Similar to the notion of quasi module for a vertex algebra, we have the following notion of quasi module for a Γ-vertex algebra:

Definition 6.3. Let V be a Γ-vertex algebra. A *quasi V-module* is a vector space W equipped with linear maps

$$Y_\alpha^W : V \to \mathrm{Hom}\,(W, W((x))) \tag{6.9}$$

for $\alpha \in \Gamma$ such that the following conditions hold:

$$Y_\alpha^W(\mathbf{1}, x) = 1_W \quad \text{for } \alpha \in \Gamma \tag{6.10}$$

and for $u, v \in V$, there exists a nonzero polynomial $f(x_1, x_2)$ such that

$$
\begin{aligned}
& x_0^{-1}\delta\left(\frac{x_1 - \beta^{-1}\alpha x_2}{x_0}\right) f(x_1, x_2) Y_\alpha^W(u, x_1) Y_\beta^W(v, x_2) \\
& -x_0^{-1}\delta\left(\frac{-\beta^{-1}\alpha x_2 + x_1}{x_0}\right) f(x_1, x_2) Y_\beta^W(v, x_2) Y_\alpha^W(u, x_1) \\
=\ & x_1^{-1}\delta\left(\frac{\beta^{-1}\alpha x_2 + x_0}{x_1}\right) f(x_1, x_2) Y_\beta^W(Y_{\beta^{-1}\alpha}(u, x_0)v, x_2)
\end{aligned}
\tag{6.11}
$$

for $\alpha, \beta \in \Gamma$. A quasi V-module W is called a *module* if (6.11) holds with $f(x_1, x_2) = 1$ for any $u, v \in V$, $\alpha, \beta \in \Gamma$.

Notice that if an (ordinary) vertex algebra V is viewed as a Γ-vertex algebra with $\Gamma = \{1\}$, the notion of quasi module for V as a Γ-vertex algebra exactly gives the notion of quasi module for V as a vertex algebra.

In terms of these notions we have:

Theorem 6.4. *Let W be a vector space, Γ a subgroup of $\mathbb{C}^\times$ and V a $\mathcal{Y}_\Gamma$-closed quasi local subspace of $\mathcal{E}(W)$ containing 1_W. Then $(U, \{\mathcal{Y}_\alpha\}, 1_W)$ carries the structure of a Γ-vertex algebra with W a quasi module with $Y_\alpha^W(a(x), x_0) = a(x_0)$ for $a(x) \in V$, $\alpha \in \Gamma$. Furthermore, for any quasi local subset S of $\mathcal{E}(W)$, $\langle S \rangle_\Gamma$ is a Γ-vertex algebra with W as a quasi module.*

Proof. Theorem 4.9 exactly states that $(V, \{\mathcal{Y}_\alpha\}, 1_W)$ carries the structure of a Γ-vertex algebra. Let $a(x), b(x) \in V$. There exists a nonzero polynomial $f(x_1, x_2)$ such that

$$f(x_1, x_2) a(x_1) b(x_2) = f(x_1, x_2) b(x_2) a(x_1). \tag{6.12}$$

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This implies that $f(x_1, x_2)a(x_1)b(x_2) \in \text{Hom}\,(W, W((x_1, x_2)))$. Let $w \in W$ and let l be a nonnegative integer such that $x^l a(x)w \in W[[x]]$, so that

$$x_1^l f(x_1, x_2)a(x_1)b(x_2)w \in W[[x_1, x_2]][x_2^{-1}].$$

For $\alpha, \beta \in \mathbb{C}^\times$, we have

$$
\begin{aligned}
f(x_1, x_2)Y_\alpha^W(a(x), x_1)Y_\beta^W(b(x), x_2) &= f(x_1, x_2)a(x_1)b(x_2) \\
&= f(x_1, x_2)b(x_2)a(x_1) \\
&= f(x_1, x_2)Y_\beta^W(b(x), x_2)Y_\alpha^W(a(x), x_1). \quad (6.13)
\end{aligned}
$$

With

$$
\begin{aligned}
Y_\alpha^W(a(x), x_0 + \beta^{-1}\alpha x_2)Y_\beta(b(x), x_2)w &= a(x_0 + \beta^{-1}\alpha x_2)b(x_2)w, \\
Y_\beta^W(\mathcal{Y}_{\beta^{-1}\alpha}(a(x), x_0)b(x), x_2)w &= (\mathcal{Y}_{\beta^{-1}\alpha}(a(x), x_0)b(x))\,w|_{x=x_2},
\end{aligned}
$$

from Proposition 3.6 we also have

$$
\begin{aligned}
&(x_0 + \beta^{-1}\alpha x_2)^l f(x_0 + \beta^{-1}\alpha x_2, x_2)Y_\alpha^W(a(x), x_0 + \beta^{-1}\alpha x_2)Y_\beta(b(x), x_2)w \\
&= (x_0 + \beta^{-1}\alpha x_2)^l f(x_0 + \beta^{-1}\alpha x_2, x_2)Y_\beta^W(\mathcal{Y}_{\beta^{-1}\alpha}(a(x), x_0)b(x), x_2)w. \quad (6.14)
\end{aligned}
$$

In view of Lemma 2.4, the required Jacobi-like identity holds. By definition $Y_\alpha^W(1_W, x_0) = 1_W$. This proves that W is a quasi module. $\qquad\square$

For the rest of this section we shall study the structures of Γ-vertex algebras and their (quasi) modules in terms of ordinary vertex algebras and modules.

Lemma 6.5. *Let $(V, \{Y_\alpha\}, \mathbf{1})$ be a Γ-vertex algebra. For $\alpha \in \Gamma$, define $R_\alpha \in \text{End}\,_{\mathbb{C}}V$ by*

$$R_\alpha(v) = \text{Res}\,_x x^{-1}Y_\alpha(v, x)\mathbf{1} = \lim_{x \to 0} Y_\alpha(v, x)\mathbf{1} \quad \text{for } v \in V. \quad (6.15)$$

Define $\mathcal{D} \in \text{End}\,_{\mathbb{C}}V$ by

$$\mathcal{D}v = \text{Res}\,_x x^{-2}Y_1(v, x)\mathbf{1} \quad \text{for } v \in V. \quad (6.16)$$

Then

$$Y_\alpha(\mathcal{D}v, x) = \frac{d}{dx}Y_\alpha(v, x), \quad (6.17)$$

$$[\mathcal{D}, Y_\alpha(v, x)] = \alpha\frac{d}{dx}Y_\alpha(v, x), \quad (6.18)$$

$$R_\alpha Y_\beta(v, x) = Y_{\alpha\beta}(v, x)R_\alpha \quad \text{for } \alpha, \beta \in \Gamma, \ v \in V. \quad (6.19)$$

Proof. Taking $\text{Res}\,_{x_1}$ of the Γ-Jacobi identity for the triple $(u, v, \mathbf{1})$ with $\beta = 1$ and using the property that $Y_\alpha(u, x_1)\mathbf{1} \in V[[x_1]]$ we get

$$
\begin{aligned}
Y_1(Y_\alpha(u, x_0)v, x_2)\mathbf{1} &= Y_\alpha(u, x_0 + \alpha x_2)Y_1(v, x_2)\mathbf{1} \\
&= e^{\alpha x_2 \frac{\partial}{\partial x_0}}Y_\alpha(u, x_0)Y_1(v, x_2)\mathbf{1}. \quad (6.20)
\end{aligned}
$$

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Then

$$\mathcal{D}Y_\alpha(u, x_0)v = \alpha \frac{\partial}{\partial x_0} Y_\alpha(u, x_0)v + Y_\alpha(u, x_0)\mathcal{D}v, \tag{6.21}$$

which shows that $[\mathcal{D}, Y_\alpha(v, x)] = \alpha \frac{d}{dx} Y_\alpha(v, x)$ for $v \in V$.

Taking Res_{x_1} of the Γ-Jacobi identity for the triple $(u, \mathbf{1}, v)$ with $\alpha = \beta$, using $Y_\alpha(\mathbf{1}, x) = 1$ we have

$$Y_\alpha(Y_1(u, x_0)\mathbf{1}, x_2)v = Y_\alpha(u, x_2 + x_0)v = e^{x_0 \frac{\partial}{\partial x_2}} Y_\alpha(u, x_2)v. \tag{6.22}$$

This gives $Y_\alpha(\mathcal{D}u, x) = \frac{d}{dx} Y_\alpha(v, x)$.

Similarly, from the Γ-Jacobi identity for the triple $(u, v, \mathbf{1})$ we have

$$Y_\beta(Y_{\beta^{-1}\alpha}(u, x_0)v, x_2)\mathbf{1} = Y_\alpha(u, x_0 + \beta^{-1}\alpha x_2)Y_\beta(v, x_2)\mathbf{1}. \tag{6.23}$$

Setting $x_2 = 0$ we get

$$R_\beta Y_{\beta^{-1}\alpha}(u, x_0)v = Y_\alpha(u, x_0)R_\beta v, \tag{6.24}$$

proving (6.19). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

In [G-K-K], Γ-conformal algebras are classified as Lie algebras equipped with an action of Γ by automorphisms of the Lie algebras. In the same spirit, the following theorem classifies Γ-vertex algebras in terms of (ordinary) vertex algebras equipped with a "compatible" Γ-action:

Theorem 6.6. *Let $(V, \{Y_\alpha\}, \mathbf{1})$ be a Γ-vertex algebra and let $R_\alpha \in \mathrm{Aut}\,_\mathbb{C}V$ for $\alpha \in \Gamma$ be defined as in Lemma 6.5. Then $(V, Y_1, \mathbf{1})$ is a vertex algebra and the map $\alpha \in \Gamma \to R_\alpha \in \mathrm{End}\,_\mathbb{C}V$ is a representation of Γ on V with $R_\alpha(\mathbf{1}) = \mathbf{1}$ for $\alpha \in \Gamma$. Furthermore,*

$$Y_\alpha(v, x) = R_\alpha Y_1(v, x)R_{\alpha^{-1}} = Y_1(R_\alpha v, \alpha^{-1}x) \quad \text{for } \alpha \in \Gamma, \ v \in V. \tag{6.25}$$

On the other hand, let V be an (ordinary) vertex algebra. Assume that V is a Γ-module with $\alpha \in \Gamma$ acting as $R_\alpha \in \mathrm{Aut}\,_\mathbb{C}V$ such that $R_\alpha(\mathbf{1}) = \mathbf{1}$ and

$$R_\alpha Y(v, x)R_\alpha^{-1} = Y(R_\alpha(v), \alpha^{-1}x) \quad \text{for } \alpha \in \Gamma, \ v \in V. \tag{6.26}$$

For $\alpha \in \Gamma$, define $Y_\alpha \in (\mathrm{End}\,V)[[x, x^{-1}]]$ by

$$Y_\alpha(v, x) = R_\alpha Y(v, x)R_\alpha^{-1} = Y(R_\alpha(v), \alpha^{-1}x) \quad \text{for } v \in V. \tag{6.27}$$

Then $(V, \{Y_\alpha\}, \mathbf{1})$ is a Γ-vertex algebra.

Proof. Let $(V, \{Y_\alpha\}, \mathbf{1})$ be a Γ-vertex algebra. Clearly, $(V, Y_1, \mathbf{1})$ is an (ordinary) vertex algebra. Using the vacuum property (6.6), the creation property (6.7) and the definition

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of R_α we immediately have that $R_\alpha(\mathbf{1}) = \mathbf{1}$ for $\alpha \in \Gamma$ and $R_1 = 1_V$. Taking Res_{x_1} of the Γ-Jacobi identity for the triple $(u, \mathbf{1}, \mathbf{1})$, we get

$$Y_\alpha(u, x_0 + \beta^{-1}\alpha x_2)Y_\beta(\mathbf{1}, x_2)\mathbf{1} = Y_\beta(Y_{\beta^{-1}\alpha}(u, x_0)\mathbf{1}, x_2)\mathbf{1}.$$

With $Y_\beta(\mathbf{1}, x) = 1$, setting $x_0 = x_2 = 0$ we obtain

$$R_\alpha(u) = R_\beta R_{\beta^{-1}\alpha}(u). \tag{6.28}$$

Thus, the map $\alpha \mapsto R_\alpha$ is a representation of Γ on V.

Taking Res_{x_1} of the Γ-Jacobi identity for the triple $(u, \mathbf{1}, v)$ with $\beta = 1$ we have

$$Y_1(Y_\alpha(u, x_0)\mathbf{1}, x_2)v = Y_\alpha(u, \alpha x_2 + x_0)v.$$

Setting $x_0 = 0$, we get
$$Y_1(R_\alpha(u), x_2)v = Y_\alpha(u, \alpha x_2)v,$$

which gives

$$Y_\alpha(u, x) = Y_1(R_\alpha(u), \alpha^{-1}x) \quad \text{for } u \in V. \tag{6.29}$$

From Lemma 6.5 we also have $R_\alpha Y_1(v, x) = Y_\alpha(v, x)R_\alpha$ for $\alpha \in \Gamma$, $v \in V$.

On the other hand, let V be a vertex algebra equipped with a Γ-module structure with α acting as $R_\alpha \in \mathrm{Aut}_{\mathbb{C}}V$ $(\alpha \in \Gamma)$ with the given properties. For $\alpha \in \Gamma$, $v \in V$, we have

$$Y_\alpha(\mathbf{1}, x) = Y(R_\alpha\mathbf{1}, \alpha^{-1}x) = Y(\mathbf{1}, \alpha^{-1}x) = 1, \tag{6.30}$$
$$Y_\alpha(v, x)\mathbf{1} = Y(R_\alpha v, \alpha^{-1}x)\mathbf{1} = e^{\alpha^{-1}x\mathcal{D}}R_\alpha v \in V[[x]], \tag{6.31}$$
$$Y_1(v, x)\mathbf{1} = e^{x\mathcal{D}}R_1 v = e^{x\mathcal{D}}v. \tag{6.32}$$

For $u, v \in V$, $\alpha, \beta \in \Gamma$, we have

$$x_0^{-1}\delta\left(\frac{\alpha^{-1}x_1 - \beta^{-1}x_2}{x_0}\right)Y(R_\alpha u, \alpha^{-1}x_1)Y(R_\beta v, \beta^{-1}x_2)$$
$$-x_0^{-1}\delta\left(\frac{-\beta^{-1}x_2 + \alpha^{-1}x_1}{x_0}\right)Y(R_\beta v, \beta^{-1}x_2)Y(R_\alpha u, \alpha^{-1}x_1)$$
$$= \alpha x_1^{-1}\delta\left(\frac{\beta^{-1}x_2 + x_0}{\alpha^{-1}x_1}\right)Y(Y(R_\alpha u, x_0)R_\beta v, \beta^{-1}x_2). \tag{6.33}$$

By rewriting the three delta functions we get

$$(\alpha x_0)^{-1}\delta\left(\frac{x_1 - \alpha\beta^{-1}x_2}{\alpha x_0}\right)Y(R_\alpha u, \alpha^{-1}x_1)Y(R_\beta v, \beta^{-1}x_2)$$
$$-(\alpha x_0)^{-1}\delta\left(\frac{-\alpha\beta^{-1}x_2 + x_1}{\alpha x_0}\right)Y(R_\beta v, \beta^{-1}x_2)Y(R_\alpha u, \alpha^{-1}x_1)$$
$$= x_1^{-1}\delta\left(\frac{\alpha\beta^{-1}x_2 + \alpha x_0}{x_1}\right)Y(Y(R_\alpha u, x_0)R_\beta v, \beta^{-1}x_2). \tag{6.34}$$

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In view of (6.26), we have

$$Y(Y(R_\alpha u, x_0)R_\beta v, \beta^{-1}x_2) = Y_\beta(Y_{\beta^{-1}\alpha}(u, \alpha x_0)v, x_2).$$

Replacing $x_0 \to \alpha^{-1}x_0$ we get the desired Γ-Jacobi identity, proving that $(V, \{Y_\alpha\}, \mathbf{1})$ is a Γ-vertex algebra. $\square$

Remark 6.7. In view of Theorem 6.6, one can define Γ-vertex algebras simply as (ordinary) vertex algebras equipped with a compatible Γ-module structure (cf. (6.26)).

In Theorem 6.6, if the group homomorphism ϕ from Γ to $\mathbb{C}^\times$ is trivial, then the second part of (6.25) and (6.26) exactly amounts to that R_α is an automorphism of vertex algebra V. In view of this, we immediately have:

Corollary 6.8. *Let Γ be any group equipped with the trivial group homomorphism from Γ to $\mathbb{C}^\times$. Then a Γ-vertex algebra exactly amounts to a vertex algebra equipped with an action of Γ by automorphisms of vertex algebra V.*

Definition 6.9. A Γ-vertex algebra V equipped with a $\mathbb{Z}$-grading $V = \coprod_{n \in \mathbb{Z}} V_{(n)}$ is called a $\mathbb{Z}$-*graded Γ-vertex algebra* if $\mathbf{1} \in V_{(0)}$ and if for any $u \in V_{(m)}$, $\alpha \in \Gamma$, $m, n, r \in \mathbb{Z}$,

$$u_{(\alpha,n)}V_{(r)} \subset V_{(r+m-n-1)}, \tag{6.35}$$

where $Y_\alpha(u, x) = \sum_{n \in \mathbb{Z}} u_{(\alpha,n)}x^{-n-1}$.

The following result states that $\mathbb{Z}$-graded Γ-vertex algebras can be characterized in terms of $\mathbb{Z}$-graded vertex algebras equipped with an action of Γ by grading-preserving automorphisms:

Proposition 6.10. *Let $V = \coprod_{n \in \mathbb{Z}} V_{(n)}$ be a $\mathbb{Z}$-graded vertex algebra in the sense that V is a vertex algebra equipped with a $\mathbb{Z}$-grading $V = \coprod_{n \in \mathbb{Z}} V_{(n)}$ such that*

$$\mathbf{1} \in V_{(0)}, \tag{6.36}$$
$$u_m V_{(n)} \subset V_{(n+k-m-1)} \quad \text{for } u \in V_{(k)}, \ k, m, n \in \mathbb{Z}. \tag{6.37}$$

Denote by $L(0)$ the degree operator, i.e., $L(0)v = nv$ for $v \in V_{(n)}$ with $n \in \mathbb{Z}$. Let π be a representation of Γ on V by grading-preserving automorphisms of V as a vertex algebra. For $\alpha \in \Gamma$, $v \in V$, set

$$Y_\alpha(v, x) = Y(\alpha^{-L(0)}\pi_\alpha v, \alpha^{-1}x) \in \text{Hom}(V, V((x))). \tag{6.38}$$

Then $(V, \{Y_\alpha\}, \mathbf{1})$ is a $\mathbb{Z}$-graded Γ-vertex algebra. On the other hand, let $(V, \{Y_\alpha\}, \mathbf{1})$ be a $\mathbb{Z}$-graded Γ-vertex algebra with the degree operator denoted by $L(0)$. For $\alpha \in \Gamma$, set

$$R_\alpha^o = \alpha^{L(0)}R_\alpha \in \text{Aut}_{\mathbb{C}}V. \tag{6.39}$$

Then R_α^o is a grading-preserving automorphism of V as a vertex algebra and R^o is a representation on V.

Proof. With (6.37), from [FHL], for $z \in \mathbb{C}^\times$, $v \in V$, we have

$$z^{L(0)} Y(v, x) z^{-L(0)} = Y(z^{L(0)}, zx), \tag{6.40}$$

so that

$$\alpha^{-L(0)} \pi_\alpha Y(v, x) \pi_\alpha^{-1} \alpha^{L(0)} = \alpha^{-L(0)} Y(\pi_\alpha v, x) \alpha^{L(0)} = Y(\alpha^{-L(0)} \pi_\alpha v, \alpha^{-1} x). \tag{6.41}$$

With $\mathbf{1} \in V_{(0)}$, we have $\alpha^{-L(0)} \pi_\alpha \mathbf{1} = \mathbf{1}$ for $\alpha \in \Gamma$. Notice that π_α for $\alpha \in \Gamma$ being grading-preserving amounts to $\alpha^{L(0)} \pi_\beta = \pi_\beta \alpha^{L(0)}$ for $\alpha, \beta \in \Gamma$, so that $\alpha \mapsto \alpha^{-L(0)} \pi_\alpha$ is a representation of Γ on V. It follows immediately from the second part of Theorem 6.6 that $(V, \{Y_\alpha\}, \mathbf{1})$ is a Γ-vertex algebra. Clearly, it is a $\mathbb{Z}$-graded Γ-vertex algebra.

On the other hand, let $(V, \{Y_\alpha\}, \mathbf{1})$ be a $\mathbb{Z}$-graded Γ-vertex algebra with $V = \coprod_{n \in \mathbb{Z}} V_{(n)}$. For $\alpha \in \Gamma$, $v \in V$, we have

$$R_\alpha^o Y_1(v, x)(R_\alpha^o)^{-1} = \alpha^{L(0)} R_\alpha Y_1(v, x) R_\alpha^{-1} \alpha^{-L(0)} = Y_1(\alpha^{L(0)} R_\alpha, x) = Y_1(R_\alpha^o v, x), \tag{6.42}$$
$$R_\alpha^o(\mathbf{1}) = \alpha^{L(0)} R_\alpha(\mathbf{1}) = \mathbf{1}, \tag{6.43}$$

using the assumption $\mathbf{1} \in V_{(0)}$. Thus, for each $\alpha \in \Gamma$, R_α^o is an automorphism of vertex algebra V. Furthermore, since V is a $\mathbb{Z}$-graded Γ-vertex algebra, for $v \in V_{(m)}$, $\alpha \in \Gamma$, we have

$$R_\alpha(v) = v_{(\alpha, -1)} \mathbf{1} \in V_{(m)},$$

recalling (6.15) for the definition of R_α. Thus R_α preserves the $\mathbb{Z}$-grading of V. Consequently, R_α commutes with $z^{L(0)}$ for any $z \in \mathbb{C}^\times$. Now, it follows that R^o is a representation of Γ on V. $\qquad\square$

As an immediate consequence we have:

Corollary 6.11. *Let $(V, Y, \mathbf{1}, \omega)$ be a vertex operator algebra and let π be a group homomorphism from Γ to $\operatorname{Aut} V$ (the automorphism group of V as a vertex operator algebra). Then $(V, \{Y_\alpha\}, \mathbf{1})$ is a Γ-vertex algebra, where*

$$Y_\alpha(v, x) = Y(\alpha^{-L(0)} \pi_\alpha(v), \alpha^{-1} x) \quad \text{for } \alpha \in \Gamma, \ v \in V. \tag{6.44}$$

The following is an analogue of a well known result in vertex algebra theory (cf. [LL]):

Proposition 6.12. *Let V be a vector space equipped with linear maps Y_α from V to $\operatorname{Hom}(V, V((x)))$ for $\alpha \in \Gamma$, a distinguished vector $\mathbf{1} \in V$, a linear operator $\mathcal{D}$ on V such that $\mathcal{D}\mathbf{1} = 0$ and a representation R of Γ on V such that $R_\alpha \mathbf{1} = \mathbf{1}$ for $\alpha \in \Gamma$ and such that*

$$Y_\alpha(\mathbf{1}, x) = 1, \tag{6.45}$$
$$Y_\alpha(v, x)\mathbf{1} \in V[[x]] \quad \text{and} \quad \lim_{x \to 0} Y_\alpha(v, x)\mathbf{1} = R_\alpha v, \tag{6.46}$$
$$[\mathcal{D}, Y_\alpha(v, x)] = \alpha \frac{d}{dx} Y_\alpha(v, x), \tag{6.47}$$
$$R_\alpha Y_\beta(v, x) = Y_{\alpha\beta}(v, x) R_\alpha \tag{6.48}$$

and for $u, v \in V$, $\alpha, \beta \in \Gamma$ there is a nonnegative integer k such that

$$(x_1 - \alpha\beta^{-1}x_2)^k [Y_\alpha(u, x_1), Y_\beta(v, x_2)] = 0. \tag{6.49}$$

Then V is a Γ-vertex algebra.

Proof. First, from [Li2], [LL] (cf. [DL], [FHL]) $(V, Y_1, \mathbf{1})$ is a vertex algebra. With the assumption (6.48), we have $Y_\alpha(v, x) = R_\alpha Y_1(v, x) R_{\alpha^{-1}}$. From (6.46) and (6.47) we have

$$Y_\alpha(u, x)\mathbf{1} = e^{\alpha^{-1}x\mathcal{D}} R_\alpha u, \tag{6.50}$$

$$e^{x\mathcal{D}} Y_\alpha(u, x_0) e^{-x\mathcal{D}} = Y_\alpha(u, x_0 + \alpha x). \tag{6.51}$$

Let $u, v \in V$ and let k be a nonnegative integer such that (6.49) holds and such that $x^k Y_\beta(v, x) R_\alpha u \in V[[x]]$, so that

$$(x_1 - \alpha\beta^{-1})^k Y_\beta(v, x_2 - \alpha^{-1}\beta x_1) u = (x_1 - \alpha\beta^{-1})^k Y_\beta(v, -\alpha^{-1}\beta x_1 + x_2) u.$$

Then we have

$$
\begin{aligned}
& (x_1 - \alpha\beta^{-1}x_2)^k Y_\alpha(u, x_1) Y_\beta(v, x_2)\mathbf{1} \\
=\ & (x_1 - \alpha\beta^{-1}x_2)^k Y_\beta(v, x_2) Y_\alpha(u, x_1)\mathbf{1} \\
=\ & (x_1 - \alpha\beta^{-1}x_2)^k Y_\beta(v, x_2) e^{\alpha^{-1}x_1\mathcal{D}} R_\alpha u \\
=\ & e^{\alpha^{-1}x_1\mathcal{D}} (x_1 - \alpha\beta^{-1}x_2)^k Y_\beta(v, x_2 - \alpha^{-1}\beta x_1) R_\alpha u \\
=\ & e^{\alpha^{-1}x_1\mathcal{D}} (x_1 - \alpha\beta^{-1}x_2)^k Y_\beta(v, -\alpha^{-1}\beta x_1 + x_2) R_\alpha u. \tag{6.52}
\end{aligned}
$$

Setting $x_2 = 0$ we get

$$x_1^k Y_\alpha(u, x_1) R_\beta v = x_1^k e^{\alpha^{-1}x_1\mathcal{D}} Y_\beta(v, -\alpha^{-1}\beta x_1) R_\alpha u,$$

which immediately gives the following skew symmetry

$$Y_\alpha(u, x_1) R_\beta v = e^{\alpha^{-1}x_1\mathcal{D}} Y_\beta(v, -\alpha^{-1}\beta x_1) R_\alpha u. \tag{6.53}$$

Taking $\beta = 1$ and then using skew symmetry for $(V, Y_1, \mathbf{1})$ we get

$$Y_\alpha(u, x_1)v = e^{\alpha^{-1}x_1\mathcal{D}} Y_1(v, -\alpha^{-1}x_1) R_\alpha u = Y_1(R_\alpha u, \alpha^{-1}x_1)v. \tag{6.54}$$

Now it follows from the second assertion of Theorem 6.6 that $(V, \{Y_\alpha\}, \mathbf{1})$ is a Γ-vertex algebra. $\square$

From the proofs of Theorem 6.6 and Proposition 6.10 we immediately have:

Proposition 6.13. *Let $(V, \{Y_\alpha\}, \mathbf{1})$ be a Γ-vertex algebra and let $(W, \{Y_\alpha^W\})$ be a quasi module for $(V, \{Y_\alpha\}, \mathbf{1})$. Then (W, Y_1^W) is a quasi module for the vertex algebra $(V, Y_1, \mathbf{1})$ and the following relation holds for $\alpha \in \Gamma$, $v \in V$, $w \in W$:*

$$Y_\alpha^W(w, x) = Y_1^W(R_\alpha v, \alpha^{-1}x). \tag{6.55}$$

On the other hand, let (W, Y_W) be a quasi module for the vertex algebra $(V, Y_1, \mathbf{1})$. For $\alpha \in \Gamma$, $v \in V$, set

$$Y_\alpha^W(v, x) = Y_W(R_\alpha v, \alpha^{-1} x). \tag{6.56}$$

Then $(W, \{Y_\alpha^W\})$ is a quasi module for the Γ-vertex algebra $(V, \{Y_\alpha\}, \mathbf{1})$.

Proof. For the first part, the first assertion is clear and the second assertion (6.55) follows from the same proof of (6.29). The assertion of the second part follows from the second part of the proof of Theorem 6.6. $\qquad\square$

7 Examples of quasi modules for vertex algebras

In this section we give three families of examples of quasi modules for vertex algebras; the first involves twisted affine Lie algebras $\hat{\mathfrak{g}}[\sigma]$ with respect to an automorphism σ of $\mathfrak{g}$ of finite order, the second involves certain quantum Heisenberg Lie algebras and the third involves certain Lie algebras on quantum torus as in [BGT].

First, following [LL] we recall the $\mathbb{N}$-graded vertex algebras associated with affine Lie algebras. In fact, all the three families of examples are related to such vertex algebras. Let $\mathfrak{g}$ be a (possibly infinite-dimensional) Lie algebra equipped with a nondegenerate symmetric invariant bilinear form $\langle \cdot, \cdot \rangle$. To the pair $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ we associate the *untwisted affine Lie algebra*, with the underlying vector space

$$\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\mathbf{k}, \tag{7.1}$$

equipped with the bracket relations

$$[a \otimes t^m, b \otimes t^n] = [a, b] \otimes t^{m+n} + m\langle a, b \rangle \delta_{m+n,0} \mathbf{k} \tag{7.2}$$

for $a, b \in \mathfrak{g}$, $m, n \in \mathbb{Z}$, together with the condition that $\mathbf{k}$ is a nonzero central element of $\hat{\mathfrak{g}}$. For $n \in \mathbb{Z}$, set

$$\hat{\mathfrak{g}}_{(0)} = \mathfrak{g} \oplus \mathbb{C}\mathbf{k}, \quad \hat{\mathfrak{g}}_{(n)} = \mathfrak{g} \otimes t^{-n} \quad \text{for } n \neq 0. \tag{7.3}$$

Then $\hat{\mathfrak{g}} = \coprod_{n \in \mathbb{Z}} \hat{\mathfrak{g}}_{(n)}$ becomes a $\mathbb{Z}$-graded Lie algebra. Set

$$\hat{\mathfrak{g}}_{(\pm)} = \coprod_{n \geq 1} \hat{\mathfrak{g}}_{(\pm n)} = \mathfrak{g} \otimes t^{\mp 1}\mathbb{C}[t^{\mp 1}], \tag{7.4}$$

which are graded Lie subalgebras of $\hat{\mathfrak{g}}$.

For $a \in \mathfrak{g}$, form the generating function

$$a(x) = \sum_{n \in \mathbb{Z}} (a \otimes t^n) x^{-n-1} \in \hat{\mathfrak{g}}[[x, x^{-1}]]. \tag{7.5}$$

For $a \in \mathfrak{g}$, $n \in \mathbb{Z}$, we use $a(n)$ for the operator on a $\hat{\mathfrak{g}}$-module, corresponding to $a \otimes t^n$. For a $\hat{\mathfrak{g}}$-module W and for $a \in \mathfrak{g}$, we set

$$a_W(x) = \sum_{n \in \mathbb{Z}} a(n) x^{-n-1} \in (\text{End } W)[[x, x^{-1}]]. \tag{7.6}$$

A $\hat{\mathfrak{g}}$-module W on which $\mathbf{k}$ acts as a scalar $\ell \in \mathbb{C}$ is said to be of *level* ℓ. A $\hat{\mathfrak{g}}$-module W is said to be *restricted* if for any $a \in \mathfrak{g}$ and $w \in W$, $a(n)w = 0$ for n sufficiently large. Then a $\hat{\mathfrak{g}}$-module W is a restricted module if and only if $a_W(x) \in \mathcal{E}(W)$ for $a \in \mathfrak{g}$.

For any complex number ℓ, let $\mathbb{C}_\ell$ be the one-dimensional module for the Lie subalgebra $\mathfrak{g} \otimes \mathbb{C}[t] \oplus \mathbb{C}\mathbf{k}$ with $\mathfrak{g} \otimes \mathbb{C}[t]$ acting trivially and with $\mathbf{k}$ acting as the scalar ℓ. Then form the induced $\hat{\mathfrak{g}}$-module

$$V_{\hat{\mathfrak{g}}}(\ell, 0) = U(\hat{\mathfrak{g}}) \otimes_{U(\mathfrak{g} \otimes \mathbb{C}[t] \oplus \mathbb{C}\mathbf{k})} \mathbb{C}_\ell, \tag{7.7}$$

which is a restricted $\hat{\mathfrak{g}}$-module of level ℓ with $1 \otimes 1$ as a generator. Set

$$\mathbf{1} = 1 \otimes 1 \in V_{\hat{\mathfrak{g}}}(\ell, 0). \tag{7.8}$$

The induced $\hat{\mathfrak{g}}$-module $V_{\hat{\mathfrak{g}}}(\ell, 0)$ is naturally an $\mathbb{N}$-graded module with $\deg \mathbb{C}_\ell = 0$. In view of the Poincare-Birkhoff-Witt theorem we have

$$V_{\hat{\mathfrak{g}}}(\ell, 0) = U(\hat{\mathfrak{g}}_{(+)}) \tag{7.9}$$

as an $\mathbb{N}$-graded vector space. Identify $\mathfrak{g}$ as a subspace of $V_{\hat{\mathfrak{g}}}(\ell, 0)$ through the linear map $a \mapsto a(-1)\mathbf{1}$.

The following results are well known (see [FZ], [Lia], cf. [Li1], [LL] for the first assertion and see [Li1], [LL], cf. [FZ] for the second assertion):

Theorem 7.1. *There exists a unique vertex algebra structure on $V_{\hat{\mathfrak{g}}}(\ell, 0)$ with $\mathbf{1}$ as vacuum vector and with $Y(a, x) = a(x)$ for $a \in \mathfrak{g}$. The vertex algebra $V_{\hat{\mathfrak{g}}}(\ell, 0)$ equipped with the grading is an $\mathbb{N}$-graded vertex algebra with $V_{\hat{\mathfrak{g}}}(\ell, 0)_{(0)} = \mathbb{C}\mathbf{1}$ and with $V_{\hat{\mathfrak{g}}}(\ell, 0)_{(1)} = \mathfrak{g}$ generates $V_{\hat{\mathfrak{g}}}(\ell, 0)$ as a vertex algebra. Furthermore, on any restricted $\hat{\mathfrak{g}}$-module W of level ℓ there exists a unique $V_{\hat{\mathfrak{g}}}(\ell, 0)$-module structure with $Y_W(a, x) = a_W(x)$ for $a \in \mathfrak{g} = V_{\hat{\mathfrak{g}}}(\ell, 0)_{(1)}$.*

Our first family of examples is about constructing quasi modules for $V_{\hat{\mathfrak{g}}}(\ell, 0)$ using restricted modules of level ℓ for a twisted affine Lie algebra.

Let σ be an automorphism of $\mathfrak{g}$ of (finite) order T, which preserves the bilinear form $\langle \cdot, \cdot \rangle$. Set

$$\omega_T = \exp\left(\frac{2\pi \sqrt{-1}}{T}\right), \tag{7.10}$$

the principal Tth root of unity, and set

$$\Gamma_T = \{\omega_T^r \mid r = 0, \ldots, T-1\}, \tag{7.11}$$

the group of Tth roots of unity. For $r \in \mathbb{Z}$, set

$$\mathfrak{g}_r = \{a \in \mathfrak{g} \mid \sigma(a) = \omega_T^r a\}. \tag{7.12}$$

Then

$$\mathfrak{g}_r = \mathfrak{g}_s \quad \text{if } r \equiv s \pmod{T}, \tag{7.13}$$

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_{T-1}. \tag{7.14}$$

Remark 7.2. It is easy to see that σ gives rise to an automorphism, also denoted by σ, of the $\mathbb{N}$-graded vertex algebra $V_{\hat{\mathfrak{g}}}(\ell, 0)$.

Associated to the triple $(\mathfrak{g}, \langle \cdot, \cdot \rangle, \sigma)$ we have a twisted affine Lie algebra (see [K1])

$$\hat{\mathfrak{g}}^\sigma = \coprod_{i=0}^{T-1} \mathfrak{g}_i \otimes t^i \mathbb{C}[t^T, t^{-T}] \oplus \mathbb{C}\mathbf{k}, \tag{7.15}$$

which is a Lie subalgebra of the untwisted affine algebra $\hat{\mathfrak{g}}$. We also have the following variant of the twisted affine Lie algebra (cf. [FLM])

$$\hat{\mathfrak{g}}[\sigma] = \coprod_{i=0}^{T-1} \mathfrak{g}_i \otimes t^{\frac{i}{T}} \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\mathbf{k}, \tag{7.16}$$

where the bracket is given by (7.2) with $m, n \in \frac{1}{T}\mathbb{Z}$. The notions of module of level ℓ and restricted module are defined in the obvious way.

Remark 7.3. It is well known (and easy to see) that the linear map defined by

$$a \otimes t^n \mapsto a \otimes t^{\frac{n}{T}}, \quad T\mathbf{k} \mapsto \mathbf{k} \tag{7.17}$$

is a Lie algebra isomorphism from $\hat{\mathfrak{g}}^\sigma$ to $\hat{\mathfrak{g}}[\sigma]$. Consequently, a (restricted) $\hat{\mathfrak{g}}^\sigma$-module of level ℓ/T exactly amounts to a (restricted) $\hat{\mathfrak{g}}[\sigma]$-module of level ℓ.

For $a \in \mathfrak{g}$, set

$$a(x)^\sigma = \sum_{i=0}^{T-1} \sigma^i(a) \otimes x^{-1} \delta \left(\frac{\omega^{-i} t}{x} \right) = \sum_{n \in \mathbb{Z}} \sum_{i=0}^{T-1} \omega_T^{-in} \left(\sigma^i(a) \otimes t^n \right) x^{-n-1} \in \hat{\mathfrak{g}}[[x, x^{-1}]]. \tag{7.18}$$

If $a \in \mathfrak{g}_s$ with $s \in \mathbb{Z}$, i.e., $\sigma(a) = \omega_T^s a$, we have

$$a(x)^\sigma = T \sum_{n \in \mathbb{Z}} (a \otimes t^{s+nT}) x^{-s-nT-1}. \tag{7.19}$$

Let $a \in \mathfrak{g}_r$, $b \in \mathfrak{g}_s$ with $r, s \in \mathbb{Z}$. Since $\langle a, b \rangle = 0$ if $r + s \notin T\mathbb{Z}$, we have

$$\langle a, b \rangle \delta_{\bar{r}+\bar{s}, \bar{0}} = \langle a, b \rangle,$$

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where $\bar{r} = r + T\mathbb{Z} \in \mathbb{Z}/T\mathbb{Z}$. Then

$$
\begin{aligned}
&[a(x_1)^\sigma, b(x_2)^\sigma]\\
=~& T^2 \sum_{m,n\in\mathbb{Z}} [a \otimes t^{r+mT}, b \otimes t^{s+nt}] x_1^{-(r+mT)-1} x_2^{-(s+nT)-1}\\
=~& T^2 \sum_{m,n\in\mathbb{Z}} \left([a,b] \otimes t^{r+s+(m+n)T} + (r+mT)\langle a,b\rangle \delta_{r+s+mT+nT,0}\mathbf{k}\right) x_1^{-(r+mT)-1} x_2^{-(s+nT)-1}\\
=~& T^2 \sum_{m,n\in\mathbb{Z}} [a,b] \otimes t^{r+s+(m+n)T} x_2^{-(r+s+mT+nT)-1} x_1^{-(r+mT)-1} x_2^{r+mT}\\
&+ T^2 \sum_{m\in\mathbb{Z}} (r+mT)\langle a,b\rangle \delta_{\bar{r}+\bar{s},\bar{0}}\mathbf{k} x_1^{-(r+mT)-1} x_2^{r+mT-1}\\
=~& T[a,b](x_2)^\sigma x_1^{-1}\delta\left(\left(\frac{x_2}{x_1}\right)^T\right)\left(\frac{x_2}{x_1}\right)^r + T^2\langle a,b\rangle \mathbf{k}\frac{\partial}{\partial x_2} x_1^{-1}\delta\left(\left(\frac{x_2}{x_1}\right)^T\right)\left(\frac{x_2}{x_1}\right)^r\\
=~& \sum_{i=0}^{T-1}\left([a,b](x_2)^\sigma \omega_T^{-ri} x_1^{-1}\delta\left(\frac{\omega_T^i x_2}{x_1}\right) + \langle a,b\rangle T\mathbf{k}\omega_T^{-ri}\frac{\partial}{\partial x_2} x_1^{-1}\delta\left(\frac{\omega_T^i x_2}{x_1}\right)\right), \qquad (7.20)
\end{aligned}
$$

where we are using the fact

$$
T x_1^{-1}\delta\left(\left(\frac{x_2}{x_1}\right)^T\right)\left(\frac{x_2}{x_1}\right)^r = \sum_{i=0}^{T-1} x_1^{-1}\delta\left(\frac{\omega_T^i x_2}{x_1}\right)\left(\frac{x_2}{x_1}\right)^r = \sum_{i=0}^{T-1} \omega_T^{-ri} x_1^{-1}\delta\left(\frac{\omega_T^i x_2}{x_1}\right). \quad (7.21)
$$

From this we have

$$
\left(\prod_{i=0}^{T-1}(x_1 - \omega_T^i x_2)^2\right)[a(x_1)^\sigma, b(x_2)^\sigma] = 0.
$$

That is,

$$
(x_1^T - x_2^T)^2[a(x_1)^\sigma, b(x_2)^\sigma] = 0. \qquad (7.22)
$$

Let W be a restricted $\hat{\mathfrak{g}}^\sigma$-module of level ℓ/T, or equivalently, a restricted $\hat{\mathfrak{g}}[\sigma]$-module of level ℓ. For $a \in \mathfrak{g}$, $a(x)^\sigma$ acts on W giving rise to an element of $\mathcal{E}(W)$, which we denote by $a_W(x)^\sigma$, Set

$$
S_W = \{a_W(x)^\sigma \mid a \in \mathfrak{g}\} \subset \mathcal{E}(W). \qquad (7.23)
$$

From (7.22), S_W is a Γ_T-local subspace. For $\alpha \in \Gamma_T$, we have

$$
R_\alpha a_W(x)^\sigma = T\sum_{n\in\mathbb{Z}} (a \otimes t^{s+nT})(\alpha x)^{-s-nT-1} = \alpha^{-s-1}a_W(x)^\sigma. \qquad (7.24)
$$

Then S_W is also stale under the action of Γ_T. By Theorems 4.9 and 5.4 S_W generates a vertex algebra $V_W = \langle S_W \rangle_{\Gamma_T}$ with W as a quasi module. By Lemma 4.12, V_W as a vertex

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algebra is generated by S_W. In view of (7.20) and Proposition 3.13 we have

$$a_W(x)^\sigma_0 b_W(x)^\sigma = [a, b]_W(x)^\sigma, \qquad (7.25)$$

$$a_W(x)^\sigma_1 b_W(x)^\sigma = \ell\langle a, b\rangle 1_W, \qquad (7.26)$$

$$a_W(x)^\sigma_n b_W(x)^\sigma = 0 \quad \text{for } n \geq 2. \qquad (7.27)$$

It follows (cf. [LL], Remark 6.2.17) that V_W is a $\hat{\mathfrak{g}}$-module of level ℓ with $a(x_1)$ acting as $\mathcal{Y}_1(a_W(x)^\sigma, x_1)$ for $a \in \mathfrak{g}$. Since S_W generates V_W as a vertex algebra, V_W is a $\hat{\mathfrak{g}}$-module generated by 1_W. It follows that there exists a $\hat{\mathfrak{g}}$-module homomorphism ψ from $V_{\hat{\mathfrak{g}}}(\ell, 0)$ onto V_W, extending the canonical map from $\mathfrak{g}$ to S_W. With $\mathfrak{g}$ as a generating subspace of $V_{\hat{\mathfrak{g}}}(\ell, 0)$, by Theorem 5.6 ψ is a vertex algebra homomorphism and W is naturally a quasi module for $V_{\hat{\mathfrak{g}}}(\ell, 0)$. Summarizing these we have:

Proposition 7.4. *Let ℓ be any complex number and let W be any restricted $\hat{\mathfrak{g}}[\sigma]$-module of level ℓ, or equivalently, a restricted $\hat{\mathfrak{g}}^\sigma$-module of level ℓ/T. Then there exists a unique quasi module structure Y_W on W for $V_{\hat{\mathfrak{g}}}(\ell, 0)$ such that $Y_W(a, x) = a_W(x)^\sigma$ for $a \in \mathfrak{g}$, with $p(x) = x^T - 1$ as the minimal polynomial.*

Remark 7.5. It was proved in [Li2] that giving a restricted $\hat{\mathfrak{g}}[\sigma]$-module structure of level ℓ on a vector space W is canonically equivalent to giving σ-twisted module structure on W for the vertex algebra $V_{\hat{\mathfrak{g}}}(\ell, 0)$. In view of this and Proposition 7.4, one naturally expects that for a general vertex operator algebra V, there should exist a canonical connection between twisted V-modules and quasi V-modules. Indeed, such a connection exists and it will be given in a sequel.

Remark 7.6. Let W be a restricted $\hat{\mathfrak{g}}$-module of level ℓ. With $\hat{\mathfrak{g}}^\sigma$ as a subalgebra by definition, W is naturally a restricted $\hat{\mathfrak{g}}^\sigma$-module of level ℓ. Then W is naturally a quasi module for $V_{\hat{\mathfrak{g}}}(\ell T, 0)$.

Now let us play a slightly different game. Fix a positive integer k. For $a \in \mathfrak{g}$, $r \in \mathbb{Z}$, set

$$E(a, r, x) = k \sum_{n \in \mathbb{Z}} (a \otimes t^{r+nk}) x^{-r-nk-1} \in \hat{\mathfrak{g}}[[x, x^{-1}]]. \qquad (7.28)$$

Clearly, if $r \equiv s \pmod{k}$ we have

$$E(a, r, x) = E(a, s, x). \qquad (7.29)$$

For $a, b \in \mathfrak{g}$, $r, s \in \mathbb{Z}$, using (7.2) we have

$$[E(a, r, x_1), E(b, s, x_2)]$$
$$= k^2 \sum_{m,n \in \mathbb{Z}} [a \otimes t^{r+mk}, b \otimes t^{s+nk}] x_1^{-r-mk-1} x_2^{-s-nk-1}$$
$$= k^2 \sum_{m,n \in \mathbb{Z}} \left([a, b] \otimes t^{r+s+(m+n)k} + (r+mk)\langle a, b\rangle \mathbf{k}\delta_{r+mk+s+nk,0}\right) x_1^{-r-mk-1} x_2^{-s-nk-1}$$
$$= \sum_{i=0}^{k-1} \left(E([a, b], r+s, x_2)\omega_k^{-ri} x_1^{-1}\delta\left(\frac{\omega_k^i x_2}{x_1}\right) + \langle a, b\rangle\delta_{\bar{r}+\bar{s},\bar{0}} k\mathbf{k}\omega_k^{-ri} \frac{\partial}{\partial x_2} x_1^{-1}\delta\left(\frac{\omega_k^i x_2}{x_1}\right) \right).$$
$$\qquad (7.30)$$

Let W be any restricted $\hat{\mathfrak{g}}$-module of level ℓ. Then $E(a, r, x) \in \mathcal{E}(W)$ for $a \in \mathfrak{g}$, $r \in \mathbb{Z}$. Set

$$E_W = \operatorname{span}\{E(a, r, x) \mid a \in \mathfrak{g}, \ r \in \mathbb{Z}\} \subset \mathcal{E}(W). \tag{7.31}$$

With (7.30), we see that E_W is a Γ_k-local subspace of $\mathcal{E}(W)$. For $\alpha \in \Gamma_k$, we have

$$R_\alpha E(a, r, x) = E(a, r, \alpha x) = \alpha^{-r-1} E(a, r, x) \quad \text{for } r \in \mathbb{Z}, \ a \in \mathfrak{g}. \tag{7.32}$$

Then E_W is closed under the action of Γ_k. By Theorems 4.9 and 5.4 E_W generates a vertex algebra $V_W = \langle E_W \rangle_{\Gamma_k}$ with W as a quasi module. By Lemma 4.12, V_W as a vertex algebra is generated by E_W. Combining (7.30) with Proposition 3.13 we have

$$E(a, r, x)_0 E(b, s, x) = E([a, b], r + s, x), \tag{7.33}$$
$$E(a, r, x)_1 E(b, s, x) = \delta_{\bar{r}+\bar{s},\bar{0}} k \langle a, b \rangle \ell, \tag{7.34}$$
$$E(a, r, x)_n E(b, s, x) = 0 \quad \text{for } n \geq 2. \tag{7.35}$$

We next relate V_W to a vertex algebra associated with an affine Lie algebra. Notice that $\mathbb{C}[\mathbb{Z}/k\mathbb{Z}]$ is a commutative associative algebra with a symmetric invariant bilinear form defined by $\langle e^{\bar{r}}, e^{\bar{s}} \rangle = \delta_{\bar{r}+\bar{s},\bar{0}}$ for $r, s \in \mathbb{Z}$. Then $\mathfrak{g} \otimes \mathbb{C}[\mathbb{Z}/k\mathbb{Z}]$ is naturally a Lie algebra which is equipped with the symmetric invariant bilinear form defined by

$$\langle a \otimes e^{\bar{r}}, b \otimes e^{\bar{s}} \rangle = \delta_{\bar{r}+\bar{s},\bar{0}} \langle a, b \rangle \quad \text{for } r, s \in \mathbb{Z}, \ a, b \in \mathfrak{g}. \tag{7.36}$$

Associated with the pair $(\mathfrak{g}[\mathbb{Z}/k\mathbb{Z}], \langle \cdot, \cdot \rangle)$ we have the following untwisted affine Lie algebra

$$\widehat{\mathfrak{g}[\mathbb{Z}/k\mathbb{Z}]} = (\mathfrak{g} \otimes \mathbb{C}[\mathbb{Z}/k\mathbb{Z}]) \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\mathbf{k}. \tag{7.37}$$

Using the same arguments for Proposition 7.4 we immediately have:

Proposition 7.7. *Let W be an (irreducible) restricted $\hat{\mathfrak{g}}$-module of level ℓ and k a positive integer. For $r \in \mathbb{Z}$, $a \in \mathfrak{g}$, set*

$$E(a, r, x) = \sum_{n \in \mathbb{Z}} a(r + nk) x^{-r-nk-1} \in \mathcal{E}(W). \tag{7.38}$$

There exists a unique (irreducible) quasi module structure on W for the vertex algebra $V_{\widehat{\mathfrak{g}[\mathbb{Z}/k\mathbb{Z}]}}(k\ell, 0)$ with $Y_W(a \otimes e^{\bar{r}}, x) = E(a, r, x)$ for $a \in \mathfrak{g}$, $r \in \mathbb{Z}$, with $x^k - 1$ as the minimal polynomial.

Remark 7.8. Since $\mathbb{C}[\mathbb{Z}/k\mathbb{Z}] = \mathbb{C}^{\otimes k}$ as a commutative associative algebra, we have that $\mathfrak{g}[\mathbb{Z}/k\mathbb{Z}] = \mathfrak{g}^{\otimes k}$. (One can find an explicit isomorphism.) Using the bilinear form one can see that $V_{\widehat{\mathfrak{g}[\mathbb{Z}/k\mathbb{Z}]}}(k\ell, 0) = V_{\hat{\mathfrak{g}}}(\ell, 0)^{\otimes k}$. In view of the first example, quasi modules for $V_{\hat{\mathfrak{g}}}(\ell, 0)^{\otimes k}$ with minimal polynomial $x^k - 1$ are closely related twisted modules. Then $V_{\hat{\mathfrak{g}}}(\ell, 0)$-modules are closely related to twisted modules for $V_{\hat{\mathfrak{g}}}(\ell, 0)^{\otimes k}$. This agrees with a general conceptual result obtained in [BDM].

Next, we present our second family of examples, which involves certain "quantum" Heisenberg Lie algebras. Let $\mathfrak{h}$ be a vector space equipped with a nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ and let q be any nonzero complex number. Consider the following Lie algebra

$$\hat{\mathfrak{h}}_q = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \tag{7.39}$$

with the following commutation relation:

$$[c, \hat{\mathfrak{h}}_q] = 0, \tag{7.40}$$
$$[a \otimes t^m, b \otimes t^n] = m\langle a, b \rangle \delta_{m+n,0}(q^m + q^{-m})c \tag{7.41}$$

for $a, b \in \mathfrak{h}$, $m, n \in \mathbb{Z}$. As with affine Lie algebras, for $a \in \mathfrak{h}$, $m \in \mathbb{Z}$, we use $a(m)$ to represent the corresponding operator of $a \otimes t^m$ on any $\hat{\mathfrak{h}}_q$-module. The notions of level and restricted module are defined in the obvious ways.

For $a \in \mathfrak{h}$, form the generating function

$$a(x) = \sum_{n \in \mathbb{Z}} a(m)x^{-m-1}. \tag{7.42}$$

In terms of the generating functions we have

$$[a(x_1), b(x_2)] = \langle a, b \rangle \frac{\partial}{\partial x_2} \left(x_1^{-1} \delta \left(\frac{qx_2}{x_1} \right) + x_1^{-1} \delta \left(\frac{q^{-1}x_2}{x_1} \right) \right) c. \tag{7.43}$$

Consequently,

$$(x_1 - qx_2)^2 (x_1 - q^{-1}x_2)^2 [a(x_1), b(x_2)] = 0. \tag{7.44}$$

For $a \in \mathfrak{h}$, $m \in \mathbb{Z}$, set

$$a(m, x) = q^m a(q^m x) \in \hat{\mathfrak{h}}_q[[x, x^{-1}]]. \tag{7.45}$$

For $a, b \in \mathfrak{h}$, $m, n \in \mathbb{Z}$, we have

$$
\begin{aligned}
[a(m, x_1), b(n, x_2)] &= q^{m+n}[a(q^m x_1), b(q^n x_2)] \\
&= \langle a, b \rangle \frac{\partial}{\partial x_2} \left(x_1^{-1} \delta \left(\frac{q^{n+1}x_2}{q^m x_1} \right) + x_1^{-1} \delta \left(\frac{q^{n-1}x_2}{q^m x_1} \right) \right) c. \quad (7.46)
\end{aligned}
$$

Set

$$\Gamma_{(q)} = \{q^m \mid m \in \mathbb{Z}\} \subset \mathbb{C}^\times. \tag{7.47}$$

Let W be any restricted $\hat{\mathfrak{h}}_q$-module of level $\ell \in \mathbb{C}$. For $a \in \mathfrak{h}$, $m \in \mathbb{Z}$, $a(m, x)$ acting on W gives rise to an element of $\mathcal{E}(W)$, which we denote by $a_W(m, x)$. Set

$$U_W = \mathrm{span}\{a_W(m, x) \mid a \in \mathfrak{h}, \ m \in \mathbb{Z}\} \subset \mathcal{E}(W). \tag{7.48}$$

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From (7.46) we see that U_W is a $\Gamma_{(q)}$-local subspace of $\mathcal{E}(W)$. Furthermore, for $a \in \mathfrak{g}$, $m, n \in \mathbb{Z}$, we have

$$R_{q^n} a_W(m, x) = a_W(m, q^n x) = q^m a_W(q^{m+n} x) = q^{-n} a_W(m+n, x). \tag{7.49}$$

Thus U_W is closed under the action of $\Gamma_{(q)}$. In view of Theorems 4.9 and 5.4 and Lemma 4.12, U_W generates a vertex algebra V_W with W as a quasi module.

Assume that q is not a root of unity, so that

$$q^m \neq q^n \quad \text{whenever } m \neq n. \tag{7.50}$$

In view of (7.46) and Proposition 3.13, the following relations hold for $a, b \in \mathfrak{h}$, $m, n \in \mathbb{Z}$:

$$a_W(m, x)_1 b_W(n, x) = \langle a, b \rangle \ell (\delta_{m,n+1} + \delta_{m,n-1}) \mathbf{1} \tag{7.51}$$
$$a_W(m, x)_r b_W(n, x) = 0 \quad \text{for } r \geq 0, \ r \neq 1. \tag{7.52}$$

On the other hand, consider the space $\mathfrak{h}[\mathbb{Z}] = \mathfrak{h} \otimes \mathbb{C}[\mathbb{Z}]$ equipped with the symmetric bilinear form $\langle \cdot, \cdot \rangle$ defined by

$$\langle a \otimes e^m, b \otimes e^n \rangle = \langle a, b \rangle (\delta_{m,n+1} + \delta_{n,m+1}) \quad \text{for } a, b \in \mathfrak{h}, \ m, n \in \mathbb{Z}. \tag{7.53}$$

Then consider the affine Lie algebra $\widehat{\mathfrak{h}[\mathbb{Z}]}$ associated with the pair $(\mathfrak{h}[\mathbb{Z}], \langle \cdot, \cdot \rangle)$, where $\mathfrak{h}[\mathbb{Z}]$ is viewed as an abelian Lie algebra. Let $V_{\widehat{\mathfrak{h}[\mathbb{Z}]}}(\ell, 0)$ be the vertex algebra associated with the affine Lie algebra $\widehat{\mathfrak{h}[\mathbb{Z}]}$ of level ℓ. This is an $\mathbb{N}$-graded vertex algebra with $V_{\widehat{\mathfrak{h}[\mathbb{Z}]}}(\ell, 0)_{(0)} = \mathbb{C}\mathbf{1}$ and $V_{\widehat{\mathfrak{h}[\mathbb{Z}]}}(\ell, 0)_{(1)} = \mathfrak{h}[\mathbb{Z}]$ as its generating subspace. Using the same arguments as we used for Proposition 7.4 we immediately have:

Proposition 7.9. *Let $\mathfrak{h}$ be a vector space equipped with a symmetric bilinear form $\langle \cdot, \cdot \rangle$, let q be a nonzero complex number which is not a root of unity and let $\hat{\mathfrak{h}}_q$ be the quantum Lie algebra defined in (7.39)–(7.41). Let $\mathfrak{h}[\mathbb{Z}]$ be the abelian Lie algebra equipped with the symmetric bilinear form defined in (7.53) and $\widehat{\mathfrak{h}[\mathbb{Z}]}$ be the corresponding affine Lie algebra. Then for any restricted $\hat{\mathfrak{h}}_q$-module W of level ℓ, there exists a unique quasi $V_{\widehat{\mathfrak{h}[\mathbb{Z}]}}(\ell, 0)$-module structure on W with $Y_W(a \otimes e^m, x) = a_W(m, x)$ for $a \in \mathfrak{g}$, $m \in \mathbb{Z}$.*

Remark 7.10. If q is a root of unity, one can appropriately modify the current analysis to work out the corresponding results.

Finally, we present our third family of examples. This family involves certain Lie algebras of quantum torus which have been studied in [BGT] and [G-K-L]. First, following [BGT] we present the Lie algebras.

Let q be a nonzero complex number. Consider the following twisted group algebra of $\mathbb{Z}^2$

$$\mathbb{C}_q[t_0^{\pm 1}, t_1^{\pm 1}] = \mathbb{C}[t_0^{\pm 1}, t_1^{\pm 1}] \tag{7.54}$$

as a vector space, where

$$t_1 t_0 = q t_0 t_1. \tag{7.55}$$

For $m, n, r, s \in \mathbb{Z}$, we have

$$(t_0^m t_1^n)(t_0^r t_1^s) = q^{nr} t_0^{m+r} t_1^{n+s}. \tag{7.56}$$

Let A be an associative algebra equipped with a nondegenerate symmetric invariant bilinear form $\langle \cdot, \cdot \rangle$ in the sense that

$$\langle ab, c \rangle = \langle a, bc \rangle \quad \text{for } a, b, c \in A. \tag{7.57}$$

Consider the tensor product associative algebra

$$A_q[t_0^{\pm 1}, t_1^{\pm 1}] = A \otimes \mathbb{C}_q[t_0^{\pm 1}, t_1^{\pm 1}]. \tag{7.58}$$

Naturally, $A_q[t_0^{\pm 1}, t_1^{\pm 1}]$ is a Lie algebra with the commutator map as the Lie bracket, which is denoted by $[\cdot, \cdot]_{loop}$. For $a, b \in A$, $m, n, r, s \in \mathbb{Z}$, we have

$$[a \otimes t_0^m t_1^n, b \otimes t_0^r t_1^s]_{loop} = q^{nr}(ab \otimes t_0^{m+r} t_1^{n+s}) - q^{ms}(ba \otimes t_0^{m+r} t_1^{n+s}). \tag{7.59}$$

We have the following two-dimensional central extension of the Lie algebra $A_q[t_0^{\pm 1}, t_1^{\pm 1}]$:

$$\hat{A}_q[t_0^{\pm 1}, t_1^{\pm 1}] = A_q[t_0^{\pm 1}, t_1^{\pm 1}] \oplus \mathbb{C}c_0 \oplus \mathbb{C}c_1 = A \otimes \mathbb{C}_q[t_0^{\pm 1}, t_1^{\pm 1}] \oplus \mathbb{C}c_0 \oplus \mathbb{C}c_1, \tag{7.60}$$

where c_0, c_1 are central and

$$[a \otimes t_0^m t_1^n, b \otimes t_0^r t_1^s] = [a \otimes t_0^m t_1^n, b \otimes t_0^r t_1^s]_{loop} + \langle a, b \rangle q^{nr} \delta_{m+r,0} \delta_{n+s,0} (mc_0 + nc_1). \tag{7.61}$$

Thus

$$\begin{aligned} &[a \otimes t_0^m t_1^n, b \otimes t_0^r t_1^s] \\ = \ & q^{nr}(ab \otimes t_0^{m+r} t_1^{n+s}) - q^{ms}(ba \otimes t_0^{m+r} t_1^{n+s}) + \langle a, b \rangle q^{nr} \delta_{m+r,0} \delta_{n+s,0} (mc_0 + nc_1) \end{aligned} \tag{7.62}$$

For $a \in A$, $n \in \mathbb{Z}$, set

$$X(a, n, x) = \sum_{m \in \mathbb{Z}} (a \otimes t_0^m t_1^n) x^{-m-1} \in \hat{A}_q[t_0^{\pm 1}, t_1^{\pm 1}][[x, x^{-1}]]. \tag{7.63}$$

Then

$$[X(a, n, x_1), X(b, s, x_2)]$$

$$= \sum_{m,r\in\mathbb{Z}} [a \otimes t_0^m t_1^n, b \otimes t_0^r t_1^s] x_1^{-m-1} x_2^{-r-1}$$

$$= \sum_{m,r\in\mathbb{Z}} \left(q^{nr}(ab \otimes t_0^{m+r} t_1^{n+s}) - q^{ms}(ba \otimes t_0^{m+r} t_1^{n+s}) \right) x_1^{-m-1} x_2^{-r-1}$$

$$+ \sum_{m,r\in\mathbb{Z}} \langle a, b\rangle q^{nr} \delta_{m+r,0} \delta_{n+s,0}(mc_0 + nc_1) x_1^{-m-1} x_2^{-r-1}$$

$$= \sum_{m,r\in\mathbb{Z}} (ab \otimes t_0^{m+r} t_1^{n+s})(q^{-n}x_2)^{-m-r-1} x_1^{-m-1}(q^{-n}x_2)^m q^{-n}$$

$$- \sum_{m,r\in\mathbb{Z}} (ba \otimes t_0^{m+r} t_1^{n+s}) x_2^{-m-r-1} x_1^{-m-1}(q^s x_2)^m$$

$$+ \sum_{m\in\mathbb{Z}} \langle a, b\rangle \delta_{n+s,0} q^{-nm}(mc_0 + nc_1) x_1^{-m-1} x_2^{m-1}$$

$$= q^{-n} X(ab, n+s, q^{-n}x_2) x_1^{-1} \delta\left(\frac{q^{-n}x_2}{x_1}\right) - X(ba, n+s, x_2) x_1^{-1} \delta\left(\frac{q^s x_2}{x_1}\right)$$

$$+ \langle a, b\rangle \delta_{n+s,0} \frac{\partial}{\partial x_2} x_1^{-1} \delta\left(\frac{q^{-n}x_2}{x_1}\right) c_0 + \langle a, b\rangle \delta_{n+s,0} n \left(c_1 x_2^{-1}\right) x_1^{-1} \delta\left(\frac{q^{-n}x_2}{x_1}\right). \quad (7.64)$$

Furthermore, for $a \in A$, $m, n \in \mathbb{Z}$, we set

$$\bar{X}(a, m, n, x) = q^n X(a, m, q^n x). \quad (7.65)$$

Then

$$[\bar{X}(a,n,m,x_1), \bar{X}(b,s,r,x_2)]$$

$$= q^{m+r}[X(a,n,q^m x_1), X(b,s,q^r x_2)]$$

$$= q^{m+r}q^{-n}X(ab, n+s, q^{-n+r}x_2)q^{-m}x_1^{-1}\delta\left(\frac{q^{-n+r}x_2}{q^m x_1}\right)$$

$$-q^{m+r}X(ba, n+s, q^r x_2)q^{-m}x_1^{-1}\delta\left(\frac{q^{s+r}x_2}{q^m x_1}\right)$$

$$+q^{m+r}\langle a,b\rangle\delta_{n+s,0}c_0 q^{-r}\frac{\partial}{\partial x_2}q^{-m}x_1^{-1}\delta\left(\frac{q^{-n+r}x_2}{q^m x_1}\right)$$

$$+q^{m+r}\langle a,b\rangle\delta_{n+s,0}c_1 q^{-r}x_2^{-1}\cdot q^{-m}x_1^{-1}\delta\left(\frac{q^{-n+r}x_2}{q^m x_1}\right)$$

$$= q^{r-n}X(ab, n+s, q^{-n+r}x_2)x_1^{-1}\delta\left(\frac{q^{r-n}x_2}{q^m x_1}\right) - q^r X(ba, n+s, q^r x_2)x_1^{-1}\delta\left(\frac{q^{s+r}x_2}{q^m x_1}\right)$$

$$+\langle a,b\rangle\delta_{n+s,0}c_0\frac{\partial}{\partial x_2}x_1^{-1}\delta\left(\frac{q^{r-n}x_2}{q^m x_1}\right) + \langle a,b\rangle\delta_{n+s,0}\left(c_1 x_2^{-1}\right)x_1^{-1}\delta\left(\frac{q^{r-n}x_2}{q^m x_1}\right)$$

$$= \bar{X}(ab, n+s, r-n, x_2)x_1^{-1}\delta\left(\frac{q^{r-n}x_2}{q^m x_1}\right) - \bar{X}(ba, n+s, r, x_2)x_1^{-1}\delta\left(\frac{q^{s+r}x_2}{q^m x_1}\right)$$

$$+\langle a,b\rangle\delta_{n+s,0}\left(c_1 x_2^{-1}\right)x_1^{-1}\delta\left(\frac{q^{r-n}x_2}{q^m x_1}\right) + \langle a,b\rangle\delta_{n+s,0}c_0\frac{\partial}{\partial x_2}x_1^{-1}\delta\left(\frac{q^{r-n}x_2}{q^m x_1}\right). \tag{7.66}$$

Set

$$c_1(x) = c_1 x^{-1}. \tag{7.67}$$

For convenience let us just use $\hat{A}_q$ for the Lie algebra $\hat{A}_q[t_0^{\pm 1}, t_1^{\pm 1}]$. An $\hat{A}_q$-module on which c_0 acts as a scalar $\ell \in \mathbb{C}$ is said to be *of level* ℓ. An $\hat{A}_q$-module W is said to be *restricted* if for any $a \in A$, $n \in \mathbb{Z}$ and for any $w \in W$, $X(a,n,x)w \in W((x))$. That is, $X(a,n,x)$ acting on W is an element of $\mathcal{E}(W)$. Denote by $X_W(a,n,x)$ the corresponding element of $\mathcal{E}(W)$ and similarly for $\bar{X}_W(a,n,m,x)$.

Let W be a restricted $\hat{A}_q$-module of level ℓ. Set

$$U_W = \text{span}\{c_1(x),\ \bar{X}_W(a,n,m,x) \mid a \in A,\ m,n \in \mathbb{Z}\} \subset \mathcal{E}(W). \tag{7.68}$$

In view of (7.66), U_W is a $\Gamma_{(q)}$-local subspace of $\mathcal{E}(W)$, where

$$\Gamma_{(q)} = \{q^m \mid m \in \mathbb{Z}\} \subset \mathbb{C}^\times.$$

We also have

$$R_{q^n}c_1(x) = c_1(x), \tag{7.69}$$

$$R_{q^r}\bar{X}_W(a,n,m,x) = \bar{X}_W(a,n,m,q^r x) = q^m X_W(a,n,q^{m+r}x)$$

$$= q^{-r}\bar{X}_W(a,n,m+r,x) \tag{7.70}$$

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for $a \in A$, $m, n, r \in \mathbb{Z}$. Then U_W is stable under the action of of $\Gamma_{(q)}$. In view of Theorem 6.4 and Lemma 4.12, U_W generates a vertex algebra V_W with W as a quasi module.

Next we determine the vertex algebra structure of V_W. The fact is that the structure of V_W depends on if q is a root of unity. Here we shall just consider the case that q is not a root of unity. That is, for any $m, n \in \mathbb{Z}$, $q^m = q^n$ if and only if $m = n$. In view of (7.66) and Proposition 3.13 we have the following relations in V_W for $a, b \in A$, $m, n, r, s \in \mathbb{Z}$:

$$\bar{X}_W(a, n, m, x)_0 \bar{X}(b, s, r, x) = \delta_{m,r-n}\bar{X}_W(ab, n+s, r-n, x)$$
$$-\delta_{m,s+r}\bar{X}_W(ba, n+s, r, x) + \langle a, b\rangle \delta_{n+s,0}\delta_{m,r-n}c_1(x), \qquad (7.71)$$
$$\bar{X}_W(a, n, m, x)_1 \bar{X}(b, s, r, x) = \langle a, b\rangle \delta_{n+s,0}\delta_{m,r-n}\ell, \qquad (7.72)$$
$$\bar{X}_W(a, n, m, x)_k \bar{X}(b, s, r, x) = 0 \qquad \text{for } k \geq 2. \qquad (7.73)$$

To determine the vertex algebra V_W, first we construct an affine Lie algebra.

Lemma 7.11. *We define a non-associative algebra with the underlying vector space*

$$\mathbb{C}_*[t_0^{\pm 1}, t_1^{\pm 1}] = \mathbb{C}[t_0^{\pm 1}, t_1^{\pm 1}], \qquad (7.74)$$

equipped with the multiplication

$$(t_0^n t_1^m)(t_0^s t_1^r) = \delta_{n+m,r} t_0^{n+s} t_1^m \qquad (7.75)$$

for $m, n, r, s \in \mathbb{Z}$. Endow this non-associative algebra with a bilinear form defined by

$$\langle t_0^n t_1^m, t_0^s t_1^r \rangle = \delta_{n+s,0}\delta_{m+n,r} \qquad (7.76)$$

for $m, n, r, s \in \mathbb{Z}$. Then this algebra is associative and the bilinear form is symmetric and associative.

Proof. This algebra is associative as

$$(t_0^k t_1^l)\left((t_0^n t_1^m)(t_0^s t_1^r)\right) = \delta_{n+m,r}(t_0^k t_1^l)(t_0^{n+s} t_1^m) = \delta_{n+m,r}\delta_{k+l,m} t_0^{k+n+s} t_1^l,$$
$$\left((t_0^k t_1^l)(t_0^n t_1^m)\right)(t_0^s t_1^r) = \delta_{k+l,m}(t_0^{k+n} t_1^l)(t_0^s t_1^r) = \delta_{k+l,m}\delta_{k+n+l,r} t_0^{k+n+s} t_1^l$$
$$= \delta_{k+l,m}\delta_{m+n,r} t_0^{k+n+s} t_1^l.$$

The bilinear form is symmetric since

$$\delta_{n+s,0}\delta_{m+n,r} = \delta_{n+s,0}\delta_{r-n,m} = \delta_{n+s,0}\delta_{r+s,m}$$

and it is invariant because

$$\langle t_0^n t_1^m, t_0^s t_1^r \rangle = \phi\left((t_0^n t_1^m)(t_0^s t_1^r)\right),$$

where ϕ is the linear functional on $\mathbb{C}_*[t_0^{\pm 1}, t_1^{\pm 1}]$ defined by

$$\phi(t_0^n t_1^m) = \delta_{n,0} \quad \text{for } m, n \in \mathbb{Z}.$$

This completes the proof. $\square$

Now, we have an associative algebra $A \otimes \mathbb{C}_*[t_0^{\pm 1}, t_1^{\pm 1}]$ equipped with a symmetric invariant bilinear form $\langle \cdot, \cdot \rangle$. Clearly, the bilinear form is also invariant when $A \otimes \mathbb{C}_*[t_0^{\pm 1}, t_1^{\pm 1}]$ is viewed as a Lie algebra. For convenience, we set

$$A_* = A \otimes \mathbb{C}_*[t_0^{\pm 1}, t_1^{\pm 1}]. \tag{7.77}$$

Form the affine Lie algebra

$$\widehat{A_*} = (A \otimes \mathbb{C}_*[t_0^{\pm 1}, t_1^{\pm 1}]) \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\mathbf{k}. \tag{7.78}$$

Denote by $V_{\widehat{A_*}}(\ell, 0)$ the $\mathbb{N}$-graded vertex algebra associated with the affine Lie algebra $\widehat{A_*}$ of level ℓ. Comparing (7.71)–(7.73) with (7.75) and (7.76) we see that V_W is an $\hat{A}_*$-module of level ℓ. Using the same arguments as we used for Proposition 7.4 (also for Proposition 7.9) we immediately have:

Proposition 7.12. *Let A be an associative algebra equipped with a symmetric invariant bilinear form $\langle \cdot, \cdot \rangle$ and let q be a nonzero complex number which is not a root of unity. Let $\hat{A}_q$ be the quantum torus Lie algebra defined in (7.60). Let A_* be the Lie algebra defined in (7.77) equipped with the symmetric invariant bilinear form. Let W be any restricted $\hat{A}_q$-module of level $\ell \in \mathbb{C}$, on which the central element c_1 acts as zero. Then there exists a unique quasi $V_{\widehat{A_*}}(\ell, 0)$-module structure on W with $Y_W(a \otimes t_0^m t_1^n, x) = \bar{X}(a, m, n, x)$ for $a \in A$, $m, n \in \mathbb{Z}$.*

Remark 7.13. One can also study the case with q a root of unity and we leave this exercise to an interested reader. If q is a primitive kth root of unity, then

$$\bar{X}(a, m, rk + n, x) = \bar{X}(a, m, n, x) \tag{7.79}$$

for $a \in A$, $m, n, r \in \mathbb{Z}$. In this case, the corresponding vertex algebra is smaller.

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